Homework 3, due Thursday October 13

In section 1.4, please do problems 9 and 10. Also, please do the following problems.

A) Show that if $u_t = D(u_{xx} + u_{yy})$, with $(x, y) \in \mathbb{R}^2$, then in terms of polar coordinates (r, θ),

$$u_t = D\left(u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}\right).$$

B) Consider the so-called bistable equation

$$u_t = u_{xx} + u(1-u)(u-\alpha), \qquad x \in \mathbb{R}, \qquad t > 0,$$

where α is a real parameter satisfying $\alpha \in (1/2, 1)$.

1. Consdier the corresponding ODE $u_t = u(1-u)(u-\alpha)$. Determine its fixed points and their stability type. (Note this explains the term "bistable".)
2. Show that, in terms of the moving coordinate $\xi = x - ct$, the PDE becomes

$$v_t = v_{\xi\xi} + cv_\xi + v(1-v)(v-\alpha). \tag{1}$$

3. Show that equilibrium solutions of this PDE, where $v_t = 0$, must satisfy the first-order ODE

$$\frac{d}{d\xi}\begin{pmatrix} v \\ w \end{pmatrix} = \begin{pmatrix} w \\ -v(1-v)(v-\alpha) - cw \end{pmatrix}$$

4. Find the fixed points of the above first order ODE and find the eigenvalues of the corresponding Jacobians to determine their stability type.
5. Show that

$$v_*(\xi) = \frac{1}{1 + e^{\frac{\xi}{\sqrt{2}}}}, \qquad c = \sqrt{2}\left(\frac{1}{2} - \alpha\right)$$

is an exact equilibrium solution, meaning it solves (1) and satisfies $v_t = 0$. Graph $v_*(\xi)$, and also draw its corresponding trajectory in the (v, w) phase plane.

In section 1.5, please do problems 3, 4, 5. For problem 4, don't worry about the part that says "Discuss your answers in the context of a vibrating guitar string. " (Although you might find it interesting to think about this.)

In section 1.7, please do problems 3, 5, and 6.

In section 1.8, please do problems 1, 6, and 8.

In section 1.9, please do problem 7.