Solutions - Midterm 1a - MA 225 - Fall 2016

Question 1 Let

$$\mathbf{u} = \langle 5, 2, -1 \rangle, \qquad \mathbf{v} = \langle -1, 0, 3 \rangle, \qquad \mathbf{w} = \langle 2, 3, 4 \rangle$$

be vectors in $\mathbb{R}^3$. For each of the following, if the quantity makes sense, compute it. If it does not make sense, explain why.

(i) **[3 points]** $\mathbf{u} \times 3\mathbf{v}$

$$\mathbf{u} \times 3\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 5 & 2 & -1 \\ -3 & 0 & 9 \end{vmatrix} = \langle 18, -42, 6 \rangle.$$

(ii) **[2 points]** $(\mathbf{u} + \mathbf{v}) \cdot \mathbf{w}$

$$(\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} = \langle 4, 2, 2 \rangle \cdot \langle 2, 3, 4 \rangle = 8 + 6 + 8 = 22.$$

(iii) **[3 points]** $\mathbf{u} \cdot (\mathbf{v} \cdot \mathbf{w})$
This does not make sense, because $\mathbf{v} \cdot \mathbf{w}$ is a scalar, and you cannot take the dot product of a scalar with the vector $\mathbf{u}$.

(iv) **[2 points]** $|-\mathbf{w}|$

$$|-\mathbf{w}| = |-1||\langle 2, 3, 4 \rangle| = \sqrt{4 + 9 + 16} = \sqrt{29}.$$

Question 2 Let

$$\mathbf{u} = \langle 2, 3, 1 \rangle, \qquad \mathbf{v} = \langle 7, 1, -1 \rangle, \qquad \mathbf{w} = \langle c, 3, 1 \rangle$$

be vectors in $\mathbb{R}^3$, where c is a scalar.

(i) **[5 points]** Find a unit vector with the same direction as $\mathbf{u}$.

$$\frac{1}{|\mathbf{u}|}\mathbf{u} = \frac{1}{\sqrt{4 + 9 + 1}}\langle 2, 3, 1 \rangle = \frac{1}{\sqrt{14}}\langle 2, 3, 1 \rangle.$$

(ii) **[5 points]** Find a value of c so that $\mathbf{w}$ is orthogonal to $\mathbf{u} \times \mathbf{v}$.
Note that any vector with the same direction as $\mathbf{u}$ or $\mathbf{v}$ will be orthogonal to $\mathbf{u} \times \mathbf{v}$. Therefore, $c = 2$ works, since then $\mathbf{w} = \mathbf{u}$.

(iii) **[5 points]** Find a value of c such that $\text{proj}_{\mathbf{u}}\mathbf{w} = \mathbf{0}$.
Since

$$\text{proj}_{\mathbf{u}}\mathbf{w} = \frac{\mathbf{u} \cdot \mathbf{w}}{|\mathbf{u}|^2}\mathbf{u}.$$

As a result, this will be the zero vector when $\mathbf{u} \cdot \mathbf{w} = 0$. Therefore, we need $0 = 2c + 9 + 1$, and so $c = -5$.

(iv) [**5 points**] Find a nonzero vector that is orthogonal to $\mathbf{u}$.
A vector $\mathbf{p} = \langle p_1, p_2, p_3 \rangle$ will be orthogonal to $\mathbf{u}$ if $\mathbf{p} \cdot \mathbf{u} = 2p_1 + 3p_2 + 1p_3 = 0$. For example, the vector $\langle -3, 2, 0 \rangle$ works.

Question 3

(i) [**5 points**] Give a geometric description of the set of points (x, y, z) that satisfy

$$x^2 + y^2 - 4y \leq 10.$$

First, notice that $y^2 - 4y = (y-2)^2 - 4$. Thus, an equivalent equation is

$$x^2 + (y-2)^2 \leq 14.$$

In the xy-plane this the set of all points that lie on or inside the circle of radius $\sqrt{14}$ centered at $(0, 2)$. Since z can be anything, in three dimensions this is the set of all point that lie inside or on the cylinder that is parallel to the z axis and that contains the circle in the xy-plane of radius $\sqrt{14}$ centered at $(0, 2)$.

(ii) [**5 points**] Write down an equation describing a plane that is orthogonal to the xy-plane and that contains the origin.
Both the xz and the yz planes are orthogonal to the xy plane and contain the origin. Their equations, respectively, are $y = 0$ and $x = 0$. (Your answer only needs to contain one of these.)

(iii) [**5 points**] Describe the set of all vectors that are parallel to $\mathbf{i} \times \mathbf{j}$ and draw a picture of the collection of all such vectors.
Note that $\mathbf{i} \times \mathbf{j} = \mathbf{k}$, which lies along the z axis. Thus, the set of all such points is the z-axis.

Question 4 [10 points]

(i) [**5 points**] Find an equation of the plane containing the points $(0, 2, 3)$, $(1, -4, 2)$, and $(1, 7, -3)$.
Two vectors that lie in the plane are $\langle 1, -6, -1 \rangle$ and $\langle 1, 5, -6 \rangle$. Therefore, we can take their cross prodcut to get a normal vector of the plane:

$$\mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -6 & -1 \\ 1 & 5 & -6 \end{vmatrix} = \langle 41, 5, 11 \rangle.$$

For a point on the plane, we can use for example $(0, 2, 3)$, which means an equation for the plane is

$$41(x-0) + 5(y-2) + 11(z-3) = 0.$$

(ii) **[5 points]** Sketch the curve described by the following function and describe in words all key aspects of your picture.

$$\mathbf{r}(t) = \langle \cos t, \sin t, t^2 \rangle, \qquad -\infty < t < \infty$$

Notice that $x^2 + y^2 = (\cos t)^2 + (\sin t)^2 = 1$. Therefore, this curve always has a "shadow" in the xy plane given by the circle of radius one centered at the origin, and it rotates counterclockwise. Since $z(t) = t^2$, we see that $z \geq 0$. Thus, this is a helix that lies along the *positive* z-axis. For increasing t when $t \leq 0$, it is rotating counterclockwise but z is decreasing, and then for t increasing and $t \geq 0$, it is still rotating counterclockwise, but now z is increasing.

Question 5 [10 points]

(i) **[7 points]** Suppose a projectile begins at the point $(1, -2, 2)$ with an initial velocity vector of $\langle 2, -1, 4 \rangle$. If its acceleration is given by

$$\mathbf{a}(t) = \langle t^2, \sin t, e^{2t} \rangle,$$

find the velocity and position vectors for $t \geq 0$.
First we compute the velocity:

$$\mathbf{v}(t) = \int \mathbf{a}(t) \mathrm{d}t + \mathbf{c}_1 = \langle \frac{1}{3}t^3, -\cos t, \frac{1}{2}e^{2t} \rangle + \mathbf{c}_1.$$

Since $\mathbf{v}(0) = \langle 0, -1, 1/2 \rangle + \mathbf{c}_1$ and we need this to equal the given initial velocity of $\langle 2, -1, 4 \rangle$, we take $\mathbf{c}_1 = \langle 2, 0, 7/2 \rangle$. As a result,

$$\mathbf{v}(t) = \langle \frac{1}{3}t^3 + 2, -\cos t, \frac{1}{2}e^{2t} + \frac{7}{2} \rangle.$$

Next, we compute the position,

$$\mathbf{r}(t) = \int \mathbf{v}(t) \mathrm{d}t + \mathbf{c}_2 = \langle \frac{1}{12}t^4 + 2t, -\sin t, \frac{1}{4}e^{2t} + \frac{7}{2}t \rangle + \mathbf{c}_2.$$

Since $\mathbf{r}(0) = \langle 0, 0, 1/4 \rangle + \mathbf{c}_2$ and we need this to equal the given initial position of $\langle 1, -2, 2 \rangle$, we take $\mathbf{c}_2 = \langle 1, -2, 7/4 \rangle$. As a result,

$$\mathbf{r}(t) = \langle \frac{1}{12}t^4 + 2t + 1, -\sin t - 2, \frac{1}{4}e^{2t} + \frac{7}{2}t + \frac{7}{4} \rangle$$

(ii) **[3 points]** Consider the motion of a projectile given by $\mathbf{r}(t) = \langle t + 5, 10 \sin t \rangle$, which is launched from the point $(5, 0)$ at $t = 0$. Where will it hit the ground?
The projectile will hit the ground when $0 = y(t) = 10 \sin t$, which means it will hit the ground when $t = \pi$. The place it will hit the ground is $x(\pi) = 5 + \pi$.

Question 6 [10 points]

(i) **[5 points]** Find the equation of the line that is orthogonal to the plane $x + 2y - z = 10$ and contains the point $(2, 7, 1)$.

If the line is orthogonal to the plane, its direction vector must be the normal vector of the plane, which is $\mathbf{v} = \langle 1, 2, -1\rangle$. Since we are told $(2, 7, 1)$ is on the line, we find that the equation of the line is

$$\mathbf{r}(t) = \langle 2, 7, 1\rangle + t\langle 1, 2, -1\rangle.$$

(ii) **[5 points]** Consider the curve $\mathbf{r}(t) = \langle e^{-t}, t^2, \cos t\rangle$. For which value(s) of t is its tangent line parallel to the x-axis?

The tangent line has direction vector $\mathbf{r}'(t) = \langle -e^{-t}, 2t, -\sin t\rangle$. The tangent line will be parallel to the x-axis when the direction vector is parallel to the x-axis, which means when it is parallel to $\langle 1, 0, 0\rangle$. This occurs when $t = 0$, since then $\mathbf{r}'(t) = \langle -1, 0, 0\rangle = -1\langle 1, 0, 0, \rangle$.

Question 7 [10 points] Consider the surface described by the equation

$$-3x^2 + \frac{y^2}{9} + z^2 = 0.$$

Sketch the traces in the three coordinate planes. Then sketch the complete surface in $\mathbb{R}^3$. Please be sure to justify your answer.

The trace in the xy plane is given by $-3x^2 + y^2/9 = 0$. This implies $y^2/9 = 3x^2$, or $y = \pm\sqrt{27}x$. These are two lines in the xy-plane that go through the origin and have slope $\pm\sqrt{27}$. (You should have included a drawing of this.) The trace in the xz plane is given by $-3x^2 + z^2 = 0$, which implies $3x^2 = z^2$, or $z = \pm\sqrt{3}x$. These are two lines in the xz plane that go through the origin and have slope $\pm\sqrt{3}$. (You should have included a drawing of this.) The trace in the yz plane is given by $y^2/9 + z^2 = 0$. This consists only of the point $(y, z) = (0, 0)$. (You should have included a drawing of this.) Putting this information together, we see that the quadratic surface is a cone that is oriented along the x axis. (You should have included a drawing of this.)

Question 8 [10 points]

(i) **[5 points]** Compute the arc length parameter for the curve $\mathbf{r}(t) = \langle t, \sin(5t), -\cos(5t)\rangle$ with base point at $t = 0$.

The arc length parameter is given by

$$s(t) = \int_0^t |\mathbf{r}'(\tau)|\mathrm{d}\tau = \int_0^t \sqrt{1 + (5\cos(5\tau))^2 + (5\sin(5\tau))^2}\mathrm{d}\tau = \sqrt{26}t.$$

(ii) **[5 points]** Consider a point $S(1, 0, c)$ for some scalar c and consider the plane $x + y + z = 5$. Find a value of c such that the distance from the point to the plane is equal to 5.

To answer this, we need a point on the plane, so for example we could use $P(5,0,0)$. We then find $\overrightarrow{PS} = \langle -4, 0, c\rangle$. The normal vector for the plane is $\mathbf{n} = \langle 1,1,1\rangle$ and therefore we need

$$5 = d = \frac{|\overrightarrow{PS} \cdot \mathbf{n}|}{|\mathbf{n}|} = \frac{|c-4|}{\sqrt{3}}.$$

As a result, either $c = 4 + 5\sqrt{3}$ or $c = 4 - 5\sqrt{3}$ work.