Solutions - Midterm 1b - MA 225 - Fall 2016

Question 1 Let

$$\mathbf{u} = \langle 3, -1, 4\rangle, \qquad \mathbf{v} = \langle 5, -1, 2\rangle, \qquad \mathbf{w} = \langle 3, -1, c\rangle$$

be vectors in $\mathbb{R}^3$, where c is a scalar.

(i) [**5 points**] Find a nonzero vector that is orthogonal to $\mathbf{u}$.
A vector $\mathbf{p} = \langle p_1, p_2, p_3\rangle$ will be orthogonal to $\mathbf{u}$ if $\mathbf{p}\cdot\mathbf{u} = 3p_1 - 1p_2 + 4p_3 = 0$. For example, the vector $\langle 1, 3, 0\rangle$ works.

(ii) [**5 points**] Find a value of c such that $\text{proj}_{\mathbf{u}}\mathbf{w} = \mathbf{0}$.
Since

$$\text{proj}_{\mathbf{u}}\mathbf{w} = \frac{\mathbf{u}\cdot\mathbf{w}}{|\mathbf{u}|^2}\mathbf{u}.$$

As a result, this will be the zero vector when $\mathbf{u}\cdot\mathbf{w} = 0$. Therefore, we need $0 = 9 + 1 + 4c$, and so $c = -10/4 = -5/2$.

(iii) [**5 points**] Find a value of c so that $\mathbf{w}$ is orthogonal to $\mathbf{u}\times\mathbf{v}$.
Note that any vector with the same direction as $\mathbf{u}$ or $\mathbf{v}$ will be orthogonal to $\mathbf{u}\times\mathbf{v}$. Therefore, $c = 4$ works, since then $\mathbf{w} = \mathbf{u}$.

(iv) [**5 points**] Find a unit vector with the same direction as $\mathbf{u}$.

$$\frac{1}{|\mathbf{u}|}\mathbf{u} = \frac{1}{\sqrt{9+1+16}}\langle 3, -1, 4\rangle = \frac{1}{\sqrt{26}}\langle 3, -1, 4\rangle.$$

Question 2 Let

$$\mathbf{u} = \langle 2, -1, 5\rangle, \qquad \mathbf{v} = \langle 0, 2, 1\rangle, \qquad \mathbf{w} = \langle 1, 2, 3\rangle$$

be vectors in $\mathbb{R}^3$. For each of the following, if the quantity makes sense, compute it. If it does not make sense, explain why.

(i) [**2 points**] $|-\mathbf{w}|$

$$|-\mathbf{w}| = |-1||\langle 1, 2, 3\rangle| = \sqrt{1+4+9} = \sqrt{14}.$$

(ii) [**3 points**] $(\mathbf{u}\cdot\mathbf{v})\cdot\mathbf{w}$
This does not make sense, because $\mathbf{u}\cdot\mathbf{v}$ is a scalar, and you cannot take the dot product of a scalar with the vector $\mathbf{w}$.

(iii) [**2 points**] $(\mathbf{u}-\mathbf{v})\cdot\mathbf{w}$

$$(\mathbf{u}-\mathbf{v})\cdot\mathbf{w} = \langle 2, -3, 4\rangle\cdot\langle 1, 2, 3\rangle = 2 - 6 + 12 = 8.$$

(iv) **[3 points]** $2\mathbf{u} \times \mathbf{v}$

$$2\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & -2 & 10 \\ 0 & 2 & 1 \end{vmatrix} = \langle -22, -4, 8 \rangle.$$

Question 3

(i) **[5 points]** Give a geometric description of the set of points (x, y, z) that satisfy

$$x^2 + y^2 + 6x \leq 10.$$

First, notice that $x^2 + 6x = (x+3)^2 - 9$. Thus, an equivalent equation is

$$(x+3)^2 + y^2 \leq 19.$$

In the xy-plane this the set of all points that lie on or inside the circle of radius $\sqrt{19}$ centered at $(-3, 0)$. Since z can be anything, in three dimensions this is the set of all point that lie inside or on the cylinder that is parallel to the z axis and that contains the circle in the xy-plane of radius $\sqrt{19}$ centered at $(-3, 0)$.

(ii) **[5 points]** Write down an equation describing a plane that is orthogonal to the yz-plane and that contains the origin.
Both the xy and the xz planes are orthogonal to the yz plane and contain the origin. Their equations, respectively, are $z = 0$ and $y = 0$. (Your answer only needs to contain one of these.)

(iii) **[5 points]** Describe the set of all vectors that are parallel to $\mathbf{j} \times \mathbf{k}$ and draw a picture of the collection of all such vectors.
Note that $\mathbf{j} \times \mathbf{k} = \mathbf{i}$, which lies along the x axis. Thus, the set of all such points is the x-axis.

Question 4 [10 points]

(i) **[7 points]** Suppose a projectile begins at the point $(1, 2, -1)$ with an initial velocity vector of $\langle 2, -1, 4 \rangle$. If its acceleration is given by

$$\mathbf{a}(t) = \langle \cos t, 2e^t, t^4 \rangle,$$

find the velocity and position vectors for $t \geq 0$.
First we compute the velocity:

$$\mathbf{v}(t) = \int \mathbf{a}(t) \mathrm{d}t + \mathbf{c}_1 = \langle \sin t, 2e^t, \frac{1}{5}t^5 \rangle + \mathbf{c}_1.$$

Since $\mathbf{v}(0) = \langle 0, 2, 0 \rangle + \mathbf{c}_1$ and we need this to equal the given initial velocity of $\langle 2, -1, 4 \rangle$, we take $\mathbf{c}_1 = \langle 2, -3, 4 \rangle$. As a result,

$$\mathbf{v}(t) = \langle \sin t + 2, 2e^t - 3, \frac{1}{5}t^5 + 4 \rangle.$$

Next, we compute the position,

$$\mathbf{r}(t) = \int \mathbf{v}(t)\mathrm{d}t + \mathbf{c}_2 = \langle -\cos t + 2t, 2e^t - 3t, \frac{1}{30}t^6 + 4t\rangle + \mathbf{c}_2.$$

Since $\mathbf{r}(0) = \langle -1, 2, 0\rangle + \mathbf{c}_2$ and we need this to equal the given initial position of $\langle 1, 2, -1\rangle$, we take $\mathbf{c}_2 = \langle 2, 0, -1\rangle$. As a result,

$$\mathbf{r}(t) = \langle -\cos t + 2t + 2, 2e^t - 3t, \frac{1}{30}t^6 + 4t - 1\rangle$$

(ii) [**3 points**] Consider the motion of a projectile given by $\mathbf{r}(t) = \langle 2t + 4, 3\sin t\rangle$, which is launched from the point $(4, 0)$ at $t = 0$. Where will it hit the ground?
The projectile will hit the ground when $0 = y(t) = 3\sin t$, which means it will hit the ground when $t = \pi$. The place it will hit the ground is $x(\pi) = 2\pi + 4$.

Question 5 [10 points]

(i) [**5 points**] Find an equation of the plane containing the points $(1, 0, -2)$, $(2, 1, -3)$, and $(0, 1, -3)$.
Two vectors that lie in the plane are $\langle 1, 1, -1\rangle$ and $\langle -1, 1, -1\rangle$. Therefore, we can take their cross prodcut to get a normal vector of the plane:

$$\mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & -1 \\ -1 & 1 & -1 \end{vmatrix} = \langle 0, 2, 2\rangle.$$

For a point on the plane, we can use for example $(1, 0, -2)$, which means an equation for the plane is

$$0(x - 1) + 2(y - 0) + 2(z + 2) = 0 \qquad \Rightarrow \qquad 2y + 2z = -4.$$

(ii) [**5 points**] Sketch the curve described by the following function and describe in words all key aspects of your picture.

$$\mathbf{r}(t) = \langle t^2, \cos t, \sin t\rangle, \qquad -\infty < t < \infty$$

Notice that $y^2 + z^2 = (\cos t)^2 + (\sin t)^2 = 1$. Therefore, this curve always has a "shadow" in the yz plane given by the circle of radius one centered at the origin, and it rotates counterclockwise. Since $x(t) = t^2$, we see that $x \geq 0$. Thus, this is a helix that lies along the *positive* x-axis. For increasing t when $t \leq 0$, it is rotating counterclockwise but x is decreasing, and then for t increasing and $t \geq 0$, it is still rotating counterclockwise, but now x is increasing.

Question 6 [10 points]

(i) [**5 points**] Compute the arc length parameter for the curve Compute the arc length parameter for the curve $\mathbf{r}(t) = \langle \cos(6t), -\sin(6t), t\rangle$ with base point at $t = 0$.
The arc length parameter is given by

$$s(t) = \int_0^t |\mathbf{r}'(\tau)|\mathrm{d}\tau = \int_0^t \sqrt{(-6\sin(5\tau))^2 + (-6\cos(5\tau))^2 + 1}\mathrm{d}\tau = \sqrt{37}t.$$

(ii) **[5 points]** Consider a point $S(c, 1, 0)$ for some scalar c and consider the plane $x + y + z = 6$. Find a value of c such that the distance from the point to the plane is equal to 8.
To answer this, we need a point on the plane, so for example we could use $P(6, 0, 0)$. We then find $\overrightarrow{PS} = \langle c - 6, 1, 0\rangle$. The normal vector for the plane is $\mathbf{n} = \langle 1, 1, 1\rangle$ and therefore we need

$$8 = d = \frac{|\overrightarrow{PS} \cdot \mathbf{n}|}{|\mathbf{n}|} = \frac{|c - 6 + 1|}{\sqrt{3}}.$$

As a result, either $c = 5 + 8\sqrt{3}$ or $c = 5 - 8\sqrt{3}$ work.

Question 7 [10 points]

(i) **[5 points]** Find the equation of the line that is orthogonal to the plane $3x - 2y + z = 8$ and contains the point $(1, -1, 2)$.
If the line is orthogonal to the plane, its direction vector must be the normal vector of the plane, which is $\mathbf{v} = \langle 3, -2, 1\rangle$. Since we are told $(1, -1, 2)$ is on the line, we find that the equation of the line is

$$\mathbf{r}(t) = \langle 1, -1, 2\rangle + t\langle 3, -2, 1\rangle.$$

(ii) **[5 points]** Consider the curve $\mathbf{r}(t) = \langle \cos t, e^{-t}, t^3\rangle$. When is its tangent line parallel to the y-axis?
The tangent line has direction vector $\mathbf{r}'(t) = \langle -\sin t, -e^{-t}, 3t^2\rangle$. The tangent line will be parallel to the y-axis when the direction vector is parallel to the y-axis, which means when it is parallel to $\langle 0, 1, 0\rangle$. This occurs when $t = 0$, since then $\mathbf{r}'(t) = \langle 0, -1, 0\rangle = -1\langle 0, 1, 0\rangle$.

Question 8 [10 points] Consider the surface described by the equation

$$6x^2 - \frac{y^2}{4} + z^2 = 0.$$

Sketch the traces in the three coordinate planes. Then sketch the complete surface in $\mathbb{R}^3$. Please be sure to justify your answer.
The trace in the xy plane is given by $6x^2 - y^2/4 = 0$. This implies $y^2/4 = 6x^2$, or $y = \pm\sqrt{24}x$. These are two lines in the xy-plane that go through the origin and have slope $\pm\sqrt{24}$. (You should have included a drawing of this.) The trace in the xz plane is given by $6x^2 + z^2 = 0$. This consists only of the point $(x, z) = (0, 0)$. (You should have included a drawing of this.) The trace in the yz plane is given by $-y^2/4 + z^2 = 0$, which implies $y^2/4 = z^2$, or $z = \pm y/2$. These are two lines in the yz plane that go through the origin and have slope $\pm 1/2$. (You should have included a drawing of this.) Putting this information together, we see that the quadratic surface is a cone that is oriented along the y axis. (You should have included a drawing of this.)