Midterm 2a – Solutions – MA 225 – Fall 2016

Question 1 [15 points]

(i) [**8 points**] Determine the domain and range of the following function. Please be sure to justify your answer.

$$f(x, y) = 5\ln(y - x^2 + 3)$$

For the domain we need $y - x^2 + 3 > 0$, or $y > x^2 - 3$. This is the set of all points in the xy-plane that lie strictly above the parabola. For the range, note that $0 < y - x^2 + 3 < \infty$. Thus, the range of this function is the range of the natural logarithm, which is $(-\infty, \infty)$.

(ii) [**7 points**] Sketch the level curves of the following function.

$$f(x, y) = 10e^{9-x^2-y^2}$$

The level curves are defined by $10e^{9-x^2-y^2} = c$, where $0 < c \leq 10e^9$. Rearranging, we find $x^2 + y^2 = 9 - \ln(c/10)$. These are circles with center the origin and radius $\sqrt{9 - \ln(c/10)}$, with $0 \leq \sqrt{9 - \ln(c/10)} < \infty$.

Question 2 [10 points]

(i) [**5 points**] Determine all values of (x, y) where the following function is continuous:

$$f(x, y) = \begin{cases} \frac{3x+y-1}{x^2+3} & (x, y) \neq (0, 0) \\ 3 & (x, y) = (0, 0) \end{cases}.$$

Be sure to justify your answer.
The function $(3x + y - 1)/(x^2 + 3)$ is continuous for all (x, y), because the denominator is never zero. Thus, the only possible problem for $f(x, y)$ is that its limit at the origin is not equal to its value at the origin. Since $(3x + y - 1)/(x^2 + 3) \to -1/3$ as $(x, y) \to (0, 0)$, which is not equal to $f(0, 0) = 3$, we find that f is continuous for all $(x, y) \neq (0, 0)$.

(ii) [**5 points**] Evaluate the following limit or determine that it does not exist.

$$\lim_{(x,y)\to(0,0)} \frac{x^2 - y}{x - y}$$

We consider lines through the origin of the form $y = mx$. Approaching along such a line, the limit becomes

$$\lim_{x\to 0} \frac{x^2 - mx}{x - mx} = \lim_{x\to 0} \frac{x - m}{1 - m} = \frac{-m}{1 - m}.$$

This is different for different values of m, so by the two-path test the limit does not exist.

Question 3 [15 points]

(i) **[8 points]** Given

$$f(x,y) = \sin\left(\frac{x}{y}\right),$$

compute $f_x(x,y)$, $f_{xx}(x,y)$, and $f_{xy}(x,y)$.

$$\begin{aligned} f_x(x,y) &= \frac{1}{y}\cos\left(\frac{x}{y}\right), \\ f_{xx} &= -\frac{1}{y^2}\sin\left(\frac{x}{y}\right) \\ f_{xy} &= -\frac{1}{y^2}\cos\left(\frac{x}{y}\right) + \frac{x}{y^3}\sin\left(\frac{x}{y}\right). \end{aligned}$$

(ii) **[7 points]** Let $g(x,y) = y\ln(2x+y)$, $x(t,s) = ts$, and $y(t,s) = e^{s^2t}$. Compute dg/ds. Make sure your answer is in terms of s and t only.

$$\begin{aligned} \frac{\partial g}{\partial s} &= \frac{\partial g}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial g}{\partial y}\frac{\partial y}{\partial s} = \frac{2y}{2x+y}(t) + \left[\ln(2x+y) + \frac{y}{2x+y}\right](2ste^{s^2t}) \\ &= \frac{2te^{s^2t}}{2st+e^{s^2t}} + \left[\ln(2st+e^{s^2t}) + \frac{e^{s^2t}}{2st+e^{s^2t}}\right]2ste^{s^2t} \end{aligned}$$

Question 4 [15 points]

(i) **[5 points]** Compute the directional derivative of the following function at the given point in the direction of the given vector.

$$h(x,y) = 37 + x^2y + \cos x, \qquad P(1,2), \qquad \langle 2,3\rangle.$$

We compute $\nabla h(x,y) = \langle 2xy - \sin x, x^2\rangle$ and $\nabla h(1,2) = \langle 4 - \sin(1), 1\rangle$. Also $\mathbf{u} = \langle 2,3\rangle/\sqrt{13}$. Hence,

$$D_{\mathbf{u}}h(1,2) = \langle 4-\sin(1),1\rangle \cdot \frac{1}{\sqrt{13}}\langle 2,3\rangle = \frac{1}{\sqrt{13}}(11 - 2\sin(1)).$$

(ii) **[5 points]** In what direction does the function $h(x,y)$ from part (i) have the greatest decrease? The greatest decrease occurs in the direction opposite to the gradient, hence in the direction of $-\langle 4-\sin(1), 1\rangle$.

(iii) **[5 points]** The derivative of a function $f(x,y)$ at a point P is greatest in the direction of the vector $\langle 1,-1\rangle$. In this direction, the value of its derivative is $3\sqrt{2}$. What is ∇f at P?
We know that the gradient points in the direction of greatest increase. Thus, $\nabla f(P) = c\langle 1,-1\rangle$ for some positive number c. Also, in that direction, the directional derivative is $|\nabla f(P)|$. Thus,

$$3\sqrt{2} = D_{\mathbf{u}}f(P) = |\nabla f(P)| = c\sqrt{2}.$$

Therefore, $c = 3$, and so $\nabla f(P) = 3\langle 1,-1\rangle$.

Question 5 [10 points]

(i) **[5 points]** Find the equation of the tangent plane to the surface $z = x^2 - xy - y^2$ at the point $(x, y) = (1, 1)$.

We have $\nabla f(x, y) = \lambda 2x - y, -x - 2y\rangle$, and so $\nabla f(1, 1) = \langle 1, -3\rangle$. Also, $f(1, 1) = -1$. Hence, the equation of the tangent plane is

$$z = -1 + 1(x - 1) - 3(y - 1)$$

(ii) **[5 points]** By about how much will $f(x, y, z) = e^x \cos(yz)$ change as the point $P(x, y, z)$ moves from the origin a distance of $ds = 0.1$ unit in the direction of $\langle 2, 2, -2\rangle$.

We use the formula $df = D_{\mathbf{u}} f(0, 0, 0)ds$, where $ds = 0.1$ and $\mathbf{u} = \langle 2, 2, -2\rangle/(2\sqrt{3})$. Since $\nabla f(x, y, z) = e^x \cos(yz), -ze^x \sin(yz), -ye^x \sin(yz)\rangle$ and $\nabla f(0, 0, 0) = \langle 1, 0, 0\rangle$. Therefore,

$$Df = \langle 1, 0, 0\rangle \cdot \frac{1}{2\sqrt{3}}\langle 2, 2, -2\rangle(0.1) = \frac{1}{\sqrt{3}}0.1 = \frac{1}{10\sqrt{3}}.$$

Question 6 [10 points] Find the local maxima, minima, and saddle points of $f(x, y) = x^3 + y^3 + 3x^2 - 3y^2 - 8$.

First, we find the critical points:

$$f_x = 3x^2 + 6x = 3x(x + 2) = 0, \qquad f_y = 3y^2 - 6y = 3y(y - 2) = 0.$$

Hence, the critical points are $(0, 0), (0, 2), (-2, 0), (-2, 2)$. To use the second derivative test, we compute

$$f_{xx} = 6x + 6, \qquad f_{xy} = 0, \qquad f_{yy} = 6y - 6.$$

Thus, $D(0, 0) = -36$, so $(0, 0)$ is a saddle; $D(0, 2) = 36$ and $f_{xx}(0, 2) = 6$, so $(0, 2)$ is a local minimum; $D(-2, 0) = 36$ and $f_{xx}(-2, 0) = -6$, so $(-2, 0)$ is a local maximum; and $D(-2, 2) = -36$, so $(-2, 2)$ is a saddle.

Question 7 [10 points]

(i) **[5 points]** Evaluate the following integral using a method of your choice.

$$\int_0^1 \int_0^{x^3} e^{y/x} \mathrm{d}y\mathrm{d}x$$

We compute

$$\int_0^1 \left[xe^{y/x}|_{y=0}^{y=x^3}\right] \mathrm{d}x = \int_0^1 \left[xe^{x^2} - x\right] \mathrm{d}x = \frac{1}{2}e^{x^2} - \frac{1}{2}x^2|_0^1 = \frac{1}{2}e - 1.$$

(ii) **[5 points]** Set up, but do not evaluate, a double integral that represents the volume of the solid that lies below the surface defined by $f(x, y) = 3xy$ and above the region in the xy-plane that lies between $y = x$, $x + y = 2$, and the x-axis. Make sure you clearly state the limits of integration and order of integration.

By drawing the region is in the xy-plane, we see that its best to integrate first with respect to x. (Otherwise you'd need two integrals. However, one can do this correctly.) Therefore, we find

$$\int_0^1 \int_y^{2-y} 3xy \mathrm{d}x \mathrm{d}y.$$

Question 8 [10 points] Consider the following integral

$$\int_0^4 \int_{-\sqrt{4-y}}^{(y-4)/2} \frac{x^3}{1+y^2} \mathrm{d}x \mathrm{d}y$$

(i) **[5 points]** Sketch the region of integration. First note that $x = (y-4)/2$ is equivalent to $y = 2x + 4$, which is a line containing the points $(-2, 0)$ and $(0, 4)$. Also, $x = -\sqrt{4-y}$ is equivalent to $y = 4 - x^2$ for $x \leq 0$. This curve also contains the points $(-2, 0)$ and $(0, 4)$, and is above the line between their two intersection points. (Your answer should have included a graph of this.)

(ii) **[5 points]** Write down an equivalent integral, with the order of integration reversed. Make sure you clearly state the limits of integration. (Note, you do not need to evaluate the resulting integral.)

$$\int_{-2}^0 \int_{2x+4}^{4-x^2} \frac{x^3}{1+y^2} \mathrm{d}y \mathrm{d}x.$$