Midterm 2b – Solutions – MA 225 – Fall 2016

Question 1 [15 points]

(i) [**8 points**] Given

$$f(x,y) = \sin\left(\frac{y}{x}\right),$$

compute $f_y(x,y)$, $f_{yy}(x,y)$, and $f_{yx}(x,y)$.

$$\begin{aligned} f_y &= \frac{1}{x}\cos\left(\frac{y}{x}\right), \\ f_{yy} &= -\frac{1}{x^2}\sin\left(\frac{y}{x}\right) \\ f_{yx} &= -\frac{1}{x^2}\cos\left(\frac{y}{x}\right) + \frac{y}{x^3}\sin\left(\frac{y}{x}\right). \end{aligned}$$

(ii) [**7 points**] Let $g(x,y) = y\ln(3x+y)$, $x(t,s) = ts$, and $y(t,s) = e^{st^2}$. Compute dg/dt. Make sure your answer is in terms of s and t only.

$$\begin{aligned} \frac{\partial g}{\partial t} &= \frac{\partial g}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial g}{\partial y}\frac{\partial y}{\partial t} = \frac{3y}{3x+y}(s) + \left[\ln(3x+y) + \frac{y}{3x+y}\right](2ste^{st^2}) \\ &= \frac{3se^{st^2}}{3st+e^{st^2}} + \left[\ln(3st+e^{st^2}) + \frac{e^{st^2}}{3st+e^{st^2}}\right] 2ste^{st^2} \end{aligned}$$

Question 2 [10 points]

(i) [**5 points**] Determine all values of (x,y) where the following function is continuous:

$$f(x,y) = \begin{cases} \frac{5x+y-1}{x^2+4} & (x,y) \neq (0,0) \\ 2 & (x,y) = (0,0) \end{cases}.$$

Be sure to justify your answer.
The function $(5x+y-1)/(x^2+4)$ is continuous for all (x,y), because the denominator is never zero. Thus, the only possible problem for $f(x,y)$ is that its limit at the origin is not equal to its value at the origin. Since $(5x+y-1)/(x^2+4) \to -1/4$ as $(x,y) \to (0,0)$, which is not equal to $f(0,0) = 2$, we find that f is continuous for all $(x,y) \neq (0,0)$.

(ii) [**5 points**] Evaluate the following limit or determine that it does not exist.

$$\lim_{(x,y)\to(0,0)} \frac{x-y^2}{x-y}$$

We consider lines through the origin of the form $y = mx$. Approaching along such a line, the limit becomes

$$\lim_{x\to 0} \frac{x - m^2x^2}{x - mx} = \lim_{x\to 0} \frac{1 - m^2x}{1 - m} = \frac{1}{1-m}.$$

This is different for different values of m, so by the two-path test the limit does not exist.

Question 3 [15 points]

(i) [**8 points**] Determine the domain and range of the following function. Please be sure to justify your answer.

$$f(x, y) = 10 \ln(y - x^2 - 2)$$

For the domain we need $y - x^2 - 2 > 0$, or $y > x^2 + 2$. This is the set of all points in the xy-plane that lie strictly above the parabola. For the range, note that $0 < y - x^2 - 3 < \infty$. Thus, the range of this function is the range of the natural logarithm, which is $(-\infty, \infty)$.

(ii) [**7 points**] Sketch the level curves of the following function.

$$f(x, y) = 5e^{4-x^2-y^2}$$

The level curves are defined by $5e^{4-x^2-y^2} = c$, where $0 < c \leq 5e^4$. Rearranging, we find $x^2 + y^2 = 4 - \ln(c/5)$. These are circles with center the origin and radius $\sqrt{4 - \ln(c/5)}$, with $0 \leq \sqrt{4 - \ln(c/5)} < \infty$.

Question 4 [15 points]

(i) [**5 points**] Compute the directional derivative of the following function at the given point in the direction of the given vector.

$$h(x, y) = 23 + xy^2 + \cos y, \qquad P(3, 1), \qquad \langle 5, 2\rangle.$$

We compute $\nabla h(x, y) = \langle y^2, 2xy - \sin y\rangle$ and $\nabla h(3, 1) = \langle 1, 6 - \sin(1)\rangle$. Also $\mathbf{u} = \langle 5, 2\rangle/\sqrt{29}$. Hence,

$$D_{\mathbf{u}}h(1, 2) = \langle 1, 6 - \sin(1)\rangle \cdot \frac{1}{\sqrt{29}}\langle 5, 2\rangle = \frac{1}{\sqrt{29}}(17 - 2\sin(1)).$$

(ii) [**5 points**] In what direction does the function $h(x, y)$ from part (i) have the greatest decrease? The greatest decrease occurs in the direction opposite to the gradient, hence in the direction of $-\langle 1, 6 - \sin(1)\rangle$.

(iii) [**5 points**] The derivative of a function $f(x, y)$ at a point P is greatest in the direction of the vector $\langle 1, -1\rangle$. In this direction, the value of its derivative is $5\sqrt{2}$. What is ∇f at P?
We know that the gradient points in the direction of greatest increase. Thus, $\nabla f(P) = c\langle 1, -1\rangle$ for some positive number c. Also, in that direction, the directional derivative is $|\nabla f(P)|$. Thus,

$$5\sqrt{2} = D_{\mathbf{u}}f(P) = |\nabla f(P)| = c\sqrt{2}.$$

Therefore, $c = 5$, and so $\nabla f(P) = 5\langle 1, -1\rangle$.

Question 5 [10 points] Find the local maxima, minima, and saddle points of $f(x, y) = x^3 + y^3 - 6x^2 + 6y^2 - 9$.
First, we find the critical points:

$$f_x = 3x^2 - 12x = 3x(x-4) = 0, \qquad f_y = 3y^2 + 12y = 3y(y+4) = 0.$$

Hence, the critical points are $(0, 0), (0, -4), (4, 0), (4, -4)$. To use the second derivative test, we compute

$$f_{xx} = 6x - 12, \qquad f_{xy} = 0, \qquad f_{yy} = 6y + 12.$$

Thus, $D(0, 0) = -144$, so $(0, 0)$ is a saddle; $D(0, -4) = 144$ and $f_{xx}(0, -4) = -12$, so $(0, -4)$ is a local maximum; $D(4, 0) = 144$ and $f_{xx}(4, 0) = 12$, so $(4, 0)$ is a local minimum; and $D(4, -4) = -144$, so $(4, -4)$ is a saddle.

Question 6 [10 points]

(i) **[5 points]** Find the equation of the tangent plane to the surface $z = y^2 + xy - x^2$ at the point $(x, y) = (1, 1)$.
We have $\nabla f(x, y) = \langle y - 2x, 2y + x\rangle$, and so $\nabla f(1, 1) = \langle -1, 3\rangle$. Also, $f(1, 1) = 1$. Hence, the equation of the tangent plane is

$$z = 1 - 1(x - 1) + 3(y - 1)$$

(ii) **[5 points]** By about how much will $f(x, y, z) = e^y \cos(xz)$ change as the point $P(x, y, z)$ moves from the origin a distance of $ds = 0.1$ unit in the direction of $\langle 3, 1, -3\rangle$.
We use the formula $df = D_{\mathbf{u}} f(0, 0, 0) ds$, where $ds = 0.1$ and $\mathbf{u} = \langle 3, 1, -3\rangle/(\sqrt{19})$. Since $\nabla f(x, y, z) = -ze^y \sin(xz), e^y \cos(xz), -xe^y \sin(xz)\rangle$ and $\nabla f(0, 0, 0) = \langle 0, 1, 0\rangle$. Therefore,

$$df = \langle 0, 1, 0\rangle \cdot \frac{1}{\sqrt{19}} \langle 3, 1, -3\rangle(0.1) = \frac{1}{\sqrt{19}} 0.1 = \frac{1}{10\sqrt{19}}.$$

Question 7 [10 points] Consider the following integral

$$\int_0^9 \int_{-\sqrt{9-y}}^{(y-9)/3} \frac{x^3}{1+y^2} \mathrm{d}x \mathrm{d}y$$

(i) **[5 points]** Sketch the region of integration. First note that $x = (y - 9)/3$ is equivalent to $y = 3x + 9$, which is a line containing the points $(-3, 0)$ and $(0, 9)$. Also, $x = -\sqrt{9 - y}$ is equivalent to $y = 9 - x^2$ for $x \leq 0$. This curve also contains the points $(-3, 0)$ and $(0, 9)$, and is above the line between their two intersection points. (Your answer should have included a graph of this.)

(ii) [**5 points**] Write down an equivalent integral, with the order of integration reversed. Make sure you clearly state the limits of integration. (Note, you do not need to evaluate the resulting integral.)

$$\int_{-3}^{0}\int_{3x+9}^{9-x^2} \frac{x^3}{1+y^2}\mathrm{d}y\mathrm{d}x.$$

Question 8 [10 points]

(i) [**5 points**] Evaluate the following integral using a method of your choice.

$$\int_0^2\int_0^{y^3} e^{x/y}\mathrm{d}x\mathrm{d}y$$

We compute

$$\int_0^2 \left[ye^{x/y}|_{x=0}^{x=y^3}\right]\mathrm{d}y = \int_0^2\left[ye^{y^2}-y\right]\mathrm{d}y = \frac{1}{2}e^{y^2}-\frac{1}{2}y^2|_0^2 = \frac{1}{2}e^4-\frac{5}{2}.$$

(ii) [**5 points**] Set up, but do not evaluate, a double integral that represents the volume of the solid that lies below the surface defined by $f(x,y) = 5xy$ and above the region in the xy-plane that lies between $y = x$, $x + y = 4$, and the x-axis. Make sure you clearly state the limits of integration and order of integration.

By drawing the region is in the xy-plane, we see that its best to integrate first with respect to x. (Otherwise you'd need two integrals. However, one can do this correctly.) Therefore, we find

$$\int_0^2\int_y^{4-y} 5xy\mathrm{d}x\mathrm{d}y.$$