Solutions to Practice Final Exam – MA 225 – Fall 2016

Question 1 Using polar coordinates, we find

$$\iint_R \sin(9x^2+9y^2)\mathrm{d}A = \int_0^{\pi/2}\int_0^4 r\sin(9r^2)\mathrm{d}r\mathrm{d}\theta = \frac{\pi}{36}(1-\cos(144)).$$

Question 2

(i) Since $\nabla \cdot \mathbf{F}$ is a scalar, it does not make sense to take the cross product of this quantity and the vector $\mathbf{b}$. Thus, it is not meaningful.

(ii) The direction vectors for the two lines are $\mathbf{v}_1 = \langle 2,3,-1\rangle$ and $\mathbf{v}_2 = \langle 4,6,-2\rangle$. These are scalar multiples of each other, $\mathbf{v}_2 = 2\mathbf{v}_1$, hence parallel. Since the direction vectors are parallel, the lines are parallel.

Question 3

$$\int_0^{\pi/2} [(\cos t\sin t)(-\sin t)+\sin t(\cos t)]\mathrm{d}t = \frac{1}{6}.$$

Question 4 We will apply the divergence theorem. The plane forming the top of the tetrahedron is $x+y+z=1$, and so

$$\iint_S \mathbf{F}\cdot\mathbf{n}\mathrm{d}\sigma \quad = \quad \iiint_D \nabla\cdot\mathbf{F}\mathrm{d}V = \int_0^1\int_0^{-x+1}\int_0^{1-x-y}(-1)\mathrm{d}z\mathrm{d}y\mathrm{d}x.$$

Question 5

(i) This is just a triple integral over the entire sphere of radius 3, and so we convert to spherical coordinates to find

$$\int_0^{2\pi}\int_0^{\pi}\int_0^3 \rho^5\sin\phi\cos\phi\mathrm{d}\rho\mathrm{d}\phi\mathrm{d}\theta = 0.$$

(ii) Since the plane doesn't intersect the xz-plane, it must be parallel to it, and hence have normal vector $\langle 0,1,0\rangle$. Thus, the plane is

$$0(x-0)+1(y-1)+0(z-0)=0 \quad \Rightarrow \quad y=1.$$

Question 6 We check all lines of the form $y=mx$:

$$\lim_{x\to 0}\frac{x^2}{3x^2+2m^2x^2} = \frac{1}{3+2m^2}$$

Since this depends on m, by the two path test the limit does not exist.

Question 7 The region of integration in the xy-plane is the region enclosed by the parabola $y=1-x^2$ in the first quadrant. Thus, the integral represents the volume of the solid region that lies under the plane $z = 1 - x$ and above the previously described region in the xy-plane. Your drawing should reflect this.

Question 8 We parameterize the surface by $\mathbf{r}(x, y) = \langle x, y, x^2 + y^2 \rangle$ where $x^2 + y^2 \leq 9$. Thus,

$$A(S) = \iint_R |\mathbf{r}_x \times \mathbf{r}_y| \mathrm{d}A = \iint_D \sqrt{1 + 4(x^2 + y^2)} \mathrm{d}A = \int_0^{2\pi} \int_0^3 r\sqrt{1 + 4r^2} \mathrm{d}r \mathrm{d}\theta = \frac{\pi}{6}[(37)^{3/2} - 1].$$

Question 9 We can think of each of these surfaces of level surfaces of functions: $F(x, y, z) = k$. The normal vector of the tangent plane of such a surface at a point (x_0, y_0, z_0) is $\nabla F(x_0, y_0, z_0)$. Thus, the normal vector for the tangent plane to the ellipsoid is $\mathbf{n}_1 = \langle 6, 4, 4 \rangle$ and the normal vector for the tangent plane to the sphere is $\mathbf{n}_2 = -\langle 6, 4, 4 \rangle$. These vectors are parallel, because they are scalar multiples of each other, and both tangent planes contain the point $(1, 1, 2)$. Since any two planes that are parallel and contain a common point must be the same plane, the ellipsoid and sphere are tangent at that point.

Question 10

(i) We have

$$f_z = xy\cos(xyz)e^{\sin(xyz)} - \frac{2x^2yz}{(1 + z^2)^2}.$$

(ii) Since $\mathbf{u} = \langle 1, 0, 1 \rangle / \sqrt{2}$ and $\nabla f(1, 2, 3) = \langle 12, 3, 2 \rangle$, we have

$$D_{\mathbf{u}}f = \langle 1, 0, 1 \rangle / \sqrt{2} \cdot \langle 12, 3, 2 \rangle = \frac{14}{\sqrt{2}}.$$