SOLUTIONS – Practice Midterm 1 – MA 225 – Fall 2016

Question 1 Let

$$\mathbf{u} = \langle 2, -1, 3\rangle, \qquad \mathbf{v} = \langle 4, 5, 0\rangle, \qquad \mathbf{w} = \langle 1, -1, -4\rangle$$

be vectors in $\mathbb{R}^3$. For each of the following, if the quantity makes sense, compute it. If it does not make sense, explain why.

(i) $(2\mathbf{u}) \cdot \mathbf{v}$

$$(2\mathbf{u}) \cdot \mathbf{v} = \langle 4, -2, 6\rangle \cdot \langle 4, 5, 0\rangle = 4(4) - 2(5) + 6(0) = 6$$

(ii) $(\mathbf{u} - \mathbf{v}) \times \mathbf{w}$

$$\begin{aligned}(\mathbf{u} - \mathbf{v}) \times \mathbf{w} &= \langle -2, -6, 3\rangle \times \langle 1, -1, -4\rangle \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & -6 & 3 \\ 1 & -1 & -4 \end{vmatrix} \\ &= (24+3)\mathbf{i} - (8-3)\mathbf{j} + (2+6)\mathbf{k} = \langle 27, -5, 8\rangle.\end{aligned}$$

(iii) $\mathbf{u} \times (\mathbf{v} \cdot \mathbf{w})$

This doesn't make sense because $\mathbf{v} \cdot \mathbf{w}$ is a scalar, not a vector, and the cross product is an operation between two vectors.

(iv) $|\mathbf{w}|$

$$|\mathbf{w}| = \sqrt{1^2 + (-1)^2 + (-4)^2} = \sqrt{18}.$$

Question 2 Let

$$\mathbf{u} = \langle 1, -5, 4\rangle, \qquad \mathbf{v} = \langle -2, c, -8\rangle, \qquad \mathbf{w} = \langle 3, 1, 0\rangle$$

be vectors in $\mathbb{R}^3$, where c is a scalar.

(i) Find two unit vectors that are parallel to $\mathbf{u}$.

$$\frac{\mathbf{u}}{|\mathbf{u}|} = \frac{1}{\sqrt{1+25+16}}\langle 1, -5, 4\rangle = \frac{1}{\sqrt{42}}\langle 1, -5, 4\rangle, \qquad -\frac{\mathbf{u}}{|\mathbf{u}|} = -\frac{1}{\sqrt{42}}\langle 1, -5, 4\rangle.$$

(ii) Find a value of c such that $\mathbf{u}$ and $\mathbf{v}$ are parallel.

Two vectors are parallel if they are scalar multiples of each other. If we take $c = 10$, then

$$\mathbf{v} = \langle -2, 10, -8\rangle = -2\langle 1, -5, 4\rangle = -2\mathbf{u}.$$

Alternatively, two vectors are parallel is their cross product is zero. So you could also take their cross product, set it equal to zero, and find the same answer.

(iii) Find a value of c such that $\mathbf{u}$ and $\mathbf{v}$ are orthogonal.
Orthogonal vectors have zero dot product, and so we need

$$0 = \mathbf{u} \cdot \mathbf{v} = -2 - 5c - 32 = -34 - 5c.$$

Therefore, we must take $c = -34/5$.

(iv) Compute $\text{proj}_{\mathbf{u}}\mathbf{w}$.

$$\text{proj}_{\mathbf{u}}\mathbf{w} = \frac{\mathbf{u} \cdot \mathbf{w}}{\mathbf{u} \cdot \mathbf{u}}\mathbf{u} = \frac{(3-5+0)}{42}\langle 1, -5, 4\rangle = -\frac{1}{21}\langle 1, -5, 4\rangle.$$

Question 3

(i) Give a geometric description of the set of points (x, y, z) that satisfy

$$x^2 + y^2 + z^2 - 4x + 6z \geq 10.$$

First, notice that

$$x^2 - 4x = (x-2)^2 - 4, \qquad z^2 + 6z = (z+3)^2 - 9,$$

and so we have

$$(x-2)^2 - 4 + y^2 + (z+3)^2 - 9 \geq 10 \qquad \Rightarrow \qquad (x-2)^2 + y^2 + (z+3)^2 \geq 23.$$

This is the set of points that lie on or outside the sphere of radius $\sqrt{23}$ with center $(2, 0, -3)$.

(ii) Write down an equation describing the plane that is parallel to the xz-plane and that contains the point $(-3, 2, -8)$.
In this plane, the x and z components can be anything, but y must be 2. Hence, the equation is

$$y = 2.$$

(iii) Describe the set of all vectors whose projection onto the unit coordinate vector $\mathbf{k}$ is zero, and draw a picture of the collection of all such vectors.
This is the set of all vectors $\mathbf{u} = \langle u_1, u_2, u_3\rangle$ such that

$$\text{proj}_{\mathbf{k}}\mathbf{u} = (\mathbf{k} \cdot \mathbf{u})\mathbf{k} = 0.$$

Therefore, $\mathbf{k}$ and $\mathbf{u}$ must be orthogonal, which means $0 = \mathbf{k} \cdot \mathbf{u} = u_3$. This is any vector of the form $\mathbf{u} = \langle u_1, u_2, 0\rangle$, which is any vector that lies in the xy-plane. (I'm not including a sketch of the xy-plane in these solutions, but it should have been in your solution.)

Question 4

(i) Find an equation of the line containing the points $(0, 2, 3)$ and $(1, -4, 2)$.
For the direction vector, one can take the vector going from the first to the second point, which is $\langle 1, -6, -1\rangle$. If one takes the first point as the point on the line, one obtains

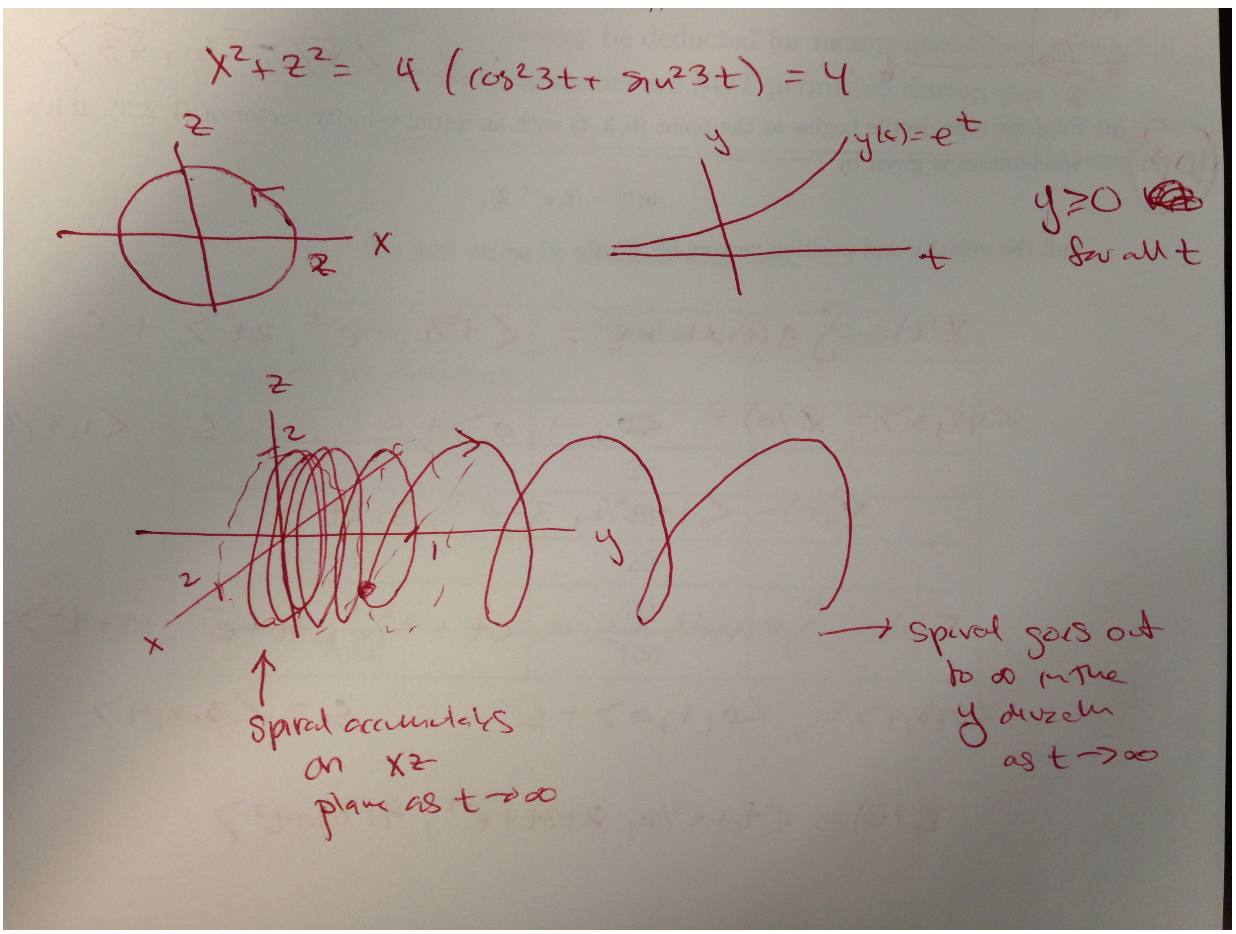
$$\mathbf{r}(t) = \langle 0, 2, 3\rangle + t\langle 1, -6, -1\rangle, \qquad -\infty < t < \infty$$

(The above is not the only correct solution to this problem.)

(ii) Sketch the curve described by the following function and describe in words all key aspects of your picture.

$$\mathbf{r}(t) = \langle 2\cos 3t, e^t, 2\sin 3t\rangle, \qquad -\infty < t < \infty$$

First, notice that $x^2 + z^2 = 4(\cos^2 3t + \sin^2 3t) = 4$, and so when projected onto the xz-plane this curve just rotates around the circle of radius 2 counterclockwise. Furthermore, since $y(t) = e^t$, we see that $y \geq 0$, y approaches zero as $t \to -\infty$, and t approaches infinity as $t \to \infty$. Putting this information together, we find



(There is a small typo in the above figure - it should say "spiral accumulates on the xz-plane as $t \to -\infty$.")

Question 5

(i) A fish in the water is climbing at an angle of 60 degrees above the horizontal with speed 2 mi/hr. Assuming the motion takes place within a two dimensional plane, find the two components of its velocity vector.

$$\mathbf{v} = 2\langle \cos 60, \sin 60\rangle = 2\langle 1/2, \sqrt{3}, 2\rangle = \langle 1, \sqrt{3}\rangle.$$

(ii) Suppose a projectile begins at the point $(0, 3, 4)$ with an initial velocity vector of $\langle 1, 2, 3\rangle$. If its acceleration is given by

$$\mathbf{a}(t) = \langle t, e^{-t}, 2\rangle,$$

find the velocity and position vectors for $t \geq 0$.

$$\mathbf{v}(t) = \int \mathbf{a}(t)\mathrm{d}t + \mathbf{c} = \langle \frac{t^2}{2}, -e^{-t}, 2t\rangle + \mathbf{c}.$$

Since, $\langle 1, 2, 3\rangle = \mathbf{v}(0) = \langle 0, -1, 0\rangle + \mathbf{c}$, we need the constant vector to be $\langle 1, 3, 3\rangle$. As a result,

$$\mathbf{v}(t) = \langle 1 + \frac{t^2}{2}, 3 - e^{-t}, 3 + 2t\rangle.$$

To compute the position,

$$\mathbf{r}(t) = \int \mathbf{v}(t)\mathrm{d}t + \mathbf{c} = \langle t + \frac{1}{6}t^3, 3t + e^{-t}, 3t + t^2\rangle + \mathbf{c}.$$

Also, $\langle 0, 3, 4\rangle = \mathbf{r}(0) = \langle 0, 1, 0\rangle + \mathbf{c}$, and so $\mathbf{c} = \langle 0, 2, 4\rangle$. Hence,

$$\mathbf{r}(t) = \langle t + \frac{1}{6}t^3, 2 + 3t + e^{-t}, 4 + 3t + t^2\rangle.$$

Question 6

(i) Find the equation of the plane that contains the following point and line.

$$(-2, 1, 4), \qquad \mathbf{r}_1(t) = \langle 1, 2, 3\rangle + t\langle -1, 0, 4\rangle, \quad -\infty < t < \infty$$

To find a normal vector of the plane, we can use two vectors that lie in the plane and take their cross product. One is the direction vector of the given line, and we can choose another to be the vector $\langle -3, -1, 1\rangle$, which is the vector between the points $(-2, 1, 4)$ and $(1, 2, 3)$, which both lie on the plane. Hence,

$$\mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3 & -1 & 1 \\ -1 & 0 & 4 \end{vmatrix} = \langle -4, 11, -1\rangle,$$

and the plane is given by $-4(x + 2) + 11(y - 1) - (z - 4) = 0$. (There is more than one correct answer. For example, one could have used the point $(1, 2, 3)$ as the point on the plane, instead of $(-2, 1, 4)$.)

(ii) Find the equation of the tangent line to the curve $\mathbf{r}(t) = \langle e^t, t^4, t^{-1}\rangle$ at the point $(e, 1, 1)$.

First note that $\mathbf{r}(1) = \langle e, 1, 1\rangle$ is a point on the line. Since $\mathbf{r}'(t) = \langle e^t, 4t^3, -t^{-2}\rangle$, we have $\mathbf{r}'(1) = \langle e, 4, -1\rangle$ is the direction vector of the line. Thus, the line is given by

$$\mathbf{R}(t) = \langle e, 1, 1\rangle + t\langle e, 4, -1\rangle.$$

Question 7 Consider the surface described by the equation

$$9z^2 + \frac{x^2}{4} - y^2 - 1 = 0$$

Sketch the traces in the three coordinate planes. Then sketch the complete surface in $\mathbb{R}^3$. Please be sure to justify your answer.

The xy-trace is given by the equation $x^2/4 - y^2 = 1$. This is a hyperbola in the xy-plane that intersects the x axis at the points $(x, y) = (\pm 2, 0)$ and does not intersect the y axis. (Your answer should include a sketch of this hyperbola.) The yz-trace is given by the equation $9z^2 - y^2 = 1$. This is a hyperbola in the yz-plane that intersects the z axis at the points $(y, z) = (0, \pm 1/3)$ and does not intersect the y axis. (Your answer should include a sketch of this hyperbola.) The xz-trace is given by the equation $9z^2 + x^2/4 = 1$. This is an ellipse in the xz-plane that intersects the z axis at the points $(x, z) = (0, \pm 1/3)$ and intersects the x axis at the points $(x, z) = (\pm 2, 0)$. (Your answer should include a sketch of this ellipse.) Putting this information together, we find a hyperboloid of one sheet that is oriented along the y-axis. (You did not necessarily need to know this was called a hyperboloid of one sheet, but your answer should have included a sketch very similar to that of the hyperboloid of one sheet in the chart on page 743 of the book, but that is oriented along the y-axis instead of the z-axis.)

Question 8

(i) Compute the length of the curve $\mathbf{r}(t) = \langle 2t^3, -t^3, 2t^3\rangle$ between the points $(2, -1, 2)$ and $(16, -8, 16)$.

First note that the two given points correspond to $t = 1$ and $t = 2$ and that $\mathbf{r}'(t) = \langle 6t^2, -3t^2, 6t^2\rangle$. As a result

$$L = \int_1^2 |\mathbf{r}'(t)|\mathrm{d}t = \int_1^2 \sqrt{(36 + 9 + 36)t^4}\mathrm{d}t = \sqrt{81}\frac{1}{3}t^3|_1^2 = \frac{7\sqrt{81}}{3}.$$

(ii) Consider a point $S(1, 0, c)$ for some scalar c and consider the line $\mathbf{r}(t) = \langle 1, 2, 0\rangle + t\langle -1, 0, 1\rangle$. Find a value of c such that the distance from the point to the line is equal to 4.

The direction vector of the line is $\mathbf{v} = \langle -1, 0, 1 \rangle$, the point $P_0(1,2,0)$ is on the line, and the vector $\overrightarrow{P_0S} = \langle 0, -2, c \rangle$. Using the formula for the distance between a point and a line, we need

$$4 = \frac{|\overrightarrow{P_0S} \times \mathbf{v}|}{|\mathbf{v}|} = \frac{|\langle -2, -c, -2 \rangle|}{\sqrt{2}} = \frac{\sqrt{8 + c^2}}{\sqrt{2}}.$$

As a result, we need $32 = 8 + c^2$, and so $c = \sqrt{24} = 2\sqrt{6}$. (Another correct answer is $c = -2\sqrt{6}$.)