Practice Midterm 2 – MA 225 – Fall 2016 - Solutions

Question 1

(i) For the domain, we need $4 - x^2 - y^2 \geq 0$, which implies $x^2 + y^2 \leq 4$. This is the set of all points on or inside the circle of radius 2 centered at the origin in the xy-plane. To determine the range, notice that $-2 \leq -\sqrt{4 - x^2 - y^2} \leq 0$, and so $e^{-2} \leq e^{-\sqrt{4-x^2-y^2}} \leq 1$. As a result, the range is $[5e^{-2}, 5]$.

(ii) The level curves are defined by the points (x, y) such that $3\ln(x - 2y^2 + 4) = c$ for any real number c. This implies $x - 2y^2 + 4 = e^{c/3}$. If we re-lable $e^{c/3} = k$, where k is any positive constant, we find that the level curves are given by $x = 2y^2 + k - 4$. These are parabolas in the xy-plane that open to the right. The value of k determines the leftmost point on the parabola. Since k must be positive, all parabolas must have their leftmost points to the right of $x = -4$. Your answer should have included a sketch of these.

Question 2

(i) True or false: if

$$\lim_{(x,0)\to(0,0)} f(x,0) = L, \qquad \lim_{(0,y)\to(0,0)} f(0,y) = L,$$

then $\lim_{(x,y)\to(0,0)} f(x,y)$ necessarily exists and is equal to L. Be sure to justify your answer.
False. In order for a limit to exist, all paths of approach to the origin must produce the same limit. The above statements only check that two specific paths (namely along the x and y axes) give the same limit. There could exist another path (like a line of the form $y = mx$) for which the limit is different.

(ii) Evaluate the following limit or determine that it does not exist.

$$\lim_{(x,y)\to(0,1)} \frac{y\sin x}{x(y+1)}$$

Recall that, using L'Hopital's rule,

$$\lim_{x\to 0} \frac{\sin x}{x} = \lim_{x\to 0} \frac{\cos x}{1} = 1.$$

Since the part of the function involving y is continuous, we can just plus in $y = 1$ and use the above calculation for x to obtain

$$\lim_{(x,y)\to(0,1)} \frac{y\sin x}{x(y+1)} = \frac{1}{1+1}(1) = \frac{1}{2}.$$

Question 3

(i) Given $f(x, y) = x^2 + y^2 - 3$, compute $f_x(1, 1)$ using the limit definition of the partial derivative.

$$f_x(1,1) = \lim_{h\to 0} \frac{f(1+h,1) - f(1,1)}{h} = \lim_{h\to 0} \frac{[(1+h)^2 + 1 - 3] - [1 + 1 - 3]}{h} = \lim_{h\to 0} \frac{2h + h^2}{h} = 2.$$

(ii) Let $g(x, y) = ye^{x^2}$, $x(t) = 2t^2$, and $y(t) = \sin t$. Compute dg/dt. Make sure your answer is in terms of t only.

$$\frac{dg}{dt} = \frac{\partial g}{\partial x}\frac{dx}{dt} + \frac{\partial g}{\partial y}\frac{dy}{dt} = 2xye^{x^2} 4t + e^{x^2} \cos t = 16t^3 \sin t e^{4t^4} + \cos t e^{4t^4}.$$

Question 4

(i) Compute the directional derivative of the following function at the given point in the direction of the given vector.

$$h(x, y) = 15 + 2x^2 - 4y^2, \qquad P(1, 4), \qquad \langle 1, 2\rangle.$$

The directional derivative is given by $D_{\mathbf{u}}h = \nabla h \cdot \mathbf{u}$, where $\mathbf{u}$ is a unit vector in the direction of the given vector. We find

$$\nabla h(1, 4) = \langle 4x, -8y\rangle|_{(1,4)} = \langle 4, -32\rangle, \qquad \mathbf{u} = \frac{1}{\sqrt{5}}\langle 1, 2\rangle,$$

and so $D_{\mathbf{u}}h(1, 4) = \langle 4, -32\rangle \cdot (1/\sqrt{5})\langle 1, 2\rangle = -60/\sqrt{5}$.

(ii) For the above function $h(x, y)$ in part (i), find a vector that points in a direction of no change in the function at $P(1, 4)$.
The function doesn't change in any direction that is orthogonal its gradient. Therefore, we need a vector that is orthogonal to $\nabla h(1, 4) = \langle 4, -32\rangle = 4\langle 1, -8\rangle$. Such a vector is given by $\langle 8, 1\rangle$ (or any scalar multiple of this vector).

(iii) For the above function $h(x, y)$ in part (i), find a vector that points in a direction of greatest change in the function at $P(1, 4)$.
The direction of greatest change is the direction of the gradient vector. Thus, the gradient vector itself works. We compute this in part (i) to be $\langle 4, -32\rangle$.

Question 5

(i) Find the normal line to the surface $x^2 - e^{xy} - y^2 \sin z = 0$ at the point $(1, 0, 0)$. Using the fact that, if $f(x, y, z) = x^2 - e^{xy} - y^2 \sin z$, we $\nabla f(x, y, z) = \langle 2x - ye^{xy}, -xe^{xy} - 2y \sin z, -y^2 \cos z\rangle$. The vector $\nabla f(1, 0, 0) = \langle 2, -1, 0\rangle$ is a direction vector for the line, and so the equation of the normal line is

$$x = 1 + 2t, \qquad y = -t, \qquad z = 0.$$

(ii) Find the linearization of $f(x,y) = e^{2y-x}$ at the point $(1,2)$.

Using the fact that $f(1,2) = e^3$, $f_x(1,2) = -e^3$, $f_y(1,2) = 2e^3$ and the formula for the linearization, we find

$$L(x,y) = f(1,2) + f_x(1,2)(x-1) + f_y(1,2)(y-2) = e^3 - e^3(x-1) + 2e^3(y-2).$$

Question 6 Use the method of Lagrange multipliers to find the dimensions of the rectangle of largest perimeter that can be inscribed in the ellipse $x^2/a^2 + y^2/b^2 = 1$ with sides parallel to the coordinate axes. What is the largest perimeter?

If we draw a rectangle inside the ellipse with upper right corner (x,y) on the ellipse, we see that the perimeter is $f(x,y) = 2x+2x+2y+2y = 4x+4y$. This is what we want to maximize. The constraint is $g(x,y) = x^2/a^2 + y^2/b^2 - 1 = 0$. We have $\nabla f(x,y) = \langle 4,4\rangle$ and $\nabla g(x,y) = \langle 2x/a^2, 2y/b^2\rangle$. The equation $\nabla f = \lambda \nabla g$ therefore gives

$$4 = \frac{2\lambda}{a^2}x, \qquad 4 = \frac{2\lambda}{b^2}y, \qquad \Rightarrow \qquad x = 2a^2/\lambda, \qquad y = 2b^2/\lambda.$$

(Note that $\lambda \neq 0$ or the above equations can't be satisfied, because $4 \neq 0$.) Plugging this into the constraint gives

$$0 = 4a^2/\lambda^2 + 4b^2/\lambda^2 - 1, \qquad \Rightarrow \qquad \lambda^2 = 4(a^2+b^2).$$

As a result,

$$(x,y) = \left(\frac{a^2}{\sqrt{a^2+b^2}}, \frac{b^2}{\sqrt{a^2+b^2}}\right),$$

the dimensions are $2a^2/\sqrt{a^2+b^2}$ by $2b^2/\sqrt{a^2+b^2}$, and the perimeter is $4\sqrt{a^2+b^2}$.

Question 7

(i) Evaluate the following integral using a method of your choice.

$$\int_0^2 \int_0^{4-y} (x+y)\mathrm{d}x\mathrm{d}y$$

$$\begin{aligned}\int_0^2 \int_0^{4-y} (x+y)\mathrm{d}x\mathrm{d}y &= \int_0^2 \left(\frac{x^2}{2} + xy\right)|_{x=0}^{x=4-y}\mathrm{d}x \\ &= \int_0^2 \left(8 - \frac{1}{2}y^2\right)\mathrm{d}y = 8y - \frac{y^3}{6}|_0^2 = 16 - \frac{4}{3} = \frac{44}{3}.\end{aligned}$$

(ii) Set up, but do not evaluate, a double integral that represents the volume of the part of the cylinder $x^2+y^2=1$ bounded above by the plane $z = 12-x-y$ and below by $z=0$. Make sure you clearly state the limits of integration and order of integration.

$$\int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (12-x-y)\mathrm{d}y\mathrm{d}x$$

(There is more than one correct answer.)

Question 8 Consider the following integral

$$\int_0^2 \int_0^{4-x^2} \frac{xe^{2y}}{4-y} \mathrm{d}y\mathrm{d}x$$

(i) Sketch the region of integration.

The parabola $y = 4 - x^2$ opens down and contains the points $(0, 4)$ and $(2, 0)$. The region is below this parabola and above the x-axis, for x between 0 and 2. (Your answer should include a sketch of this.)

(ii) Reverse the order of integration. (Note, you do not need to evaluate the resulting integral.)

$$\int_0^4 \int_0^{\sqrt{4-y}} \frac{xe^{2y}}{4-y} \mathrm{d}x\mathrm{d}y$$