Midterm Solutions – MA 561 – Fall 2016

Question 1 [10 points] For each of the following PDEs

(i) $u_t - u_{xx} + x^2 = 0$

(ii) $u_x + e^x u_y = 0$

(iii) $u_x(1 + u_x^2)^{-1} + u_y(1 + u_y^2)^{-1} = 0$

answer the following questions:

(a) Is it linear or nonlinear?

(b) Is it homogeneous or inhomogeneous?

(c) What is its order?

Please be sure to give reasons for your answer.

(i) is linear (not nonlinear terms in u), inhomogeneous (because of the x^2 term), and second order (because the highest derivative is u_{xx}. (ii) is linear (same reason as i), homogeneous (no terms that don't involve u), and first order (because the highest derivative is u_x and u_y). (iii) is nonlinear (because, for example, of the u_x^2 term), homogeneous (same reason ii), and first order (same reason as ii).

Question 2 [10 points] Suppose both u_1 and u_2 are solutions of the PDE $\mathcal{L}u = g$, where $\mathcal{L}$ is some linear operator and g is a given function. Show that $w = u_1 - u_2$ is a solution of $\mathcal{L}w = 0$.

Notice that $\mathcal{L}w = \mathcal{L}(u_1 - u_2) = \mathcal{L}u_1 - \mathcal{L}u_2$ by linearity of the operator $\mathcal{L}$. Since, $\mathcal{L}u_{1,2} = g$, we therefore find that $\mathcal{L}w = g - g = 0$.

Question 3 [10 points] Verify that $u(x, y) = f(x)g(y)$ is a solution of $uu_{xy} = u_x u_y$ for any given differentiable functions f and g

We can compute $u_x = f'(x)g(y)$, $u_y = f(x)g'(y)$ and $u_{xy} = f'(x)g'(y)$. Therefore, $uu_{xy} = f(x)g(y)f'(x)g'(y)$ and $u_x u_y = f'(x)g(y)f(x)g'(y)$, which are equal.

Question 4 [10 points]

Verify that the function

$$u(x,t) = \frac{x}{t^{3/2}} e^{-\frac{x^2}{4t}}$$

is a solution to the heat equation $u_t = u_{xx}$.

Directly computing the derivatives we find

$$u_x = \frac{1}{t^{3/2}} e^{-\frac{x^2}{4t}} - \frac{x^2}{2t^{5/2}} e^{-\frac{x^2}{4t}}, \qquad u_{xx} = -\frac{3x}{2t^{5/2}} e^{-\frac{x^2}{4t}} + \frac{x^3}{4t^{7/2}} e^{-\frac{x^2}{4t}}, \qquad u_t = -\frac{3x}{2t^{5/2}} e^{-\frac{x^2}{4t}} + \frac{x^3}{4t^{7/2}} e^{-\frac{x^2}{4t}}.$$

Therefore, we see that $u_t = u_{xx}$.

Question 5 [10 points] Consider the equation

$$\begin{aligned} 2u_t + 3u_x = 0, \qquad x \in \mathbb{R}, \qquad t > 0 \\ u(x,0) = \sin x \qquad x \in \mathbb{R}. \end{aligned}$$

(i) Find an equation for the characteristics and sketch them in the (x,t) plane. (An equivalent equation is $u_t + (3/2)u_x = 0$, so $c = 3/2$ and the characteristics are $x - (3/2)t = \xi$ for any $\xi \in \mathbb{R}$. (You could equivalently derive the equation for the characteristics by differentiating $u(x(t;\xi),t)$ with respect to t and setting it equal to zero.) These lines in the (x,t) plane with slope $2/3$ that intersect the x-axis at $x = \xi$. (You should have included a picture.)

(ii) Write down an explicit formula for the solution. We have $u(x(t;\xi),t) = u(x(0;\xi),0) = u(\xi,0) = \sin\xi$, because the solution is constant on characteristics. Therefore, $u(x,t) = \sin(x - (3/2)t)$.

(iii) Sketch the solution for $t = 0$ and describe how it behaves for $t > 0$. For $t = 0$, $u(x,0) = \sin x$ is just a sine wave. (You should have included a picture.) As t increases, this sine wave rigidly moves to the right with speed $3/2$.

Question 6 [10 points] Solve the equation

$$\begin{aligned} u_t + t^2 u_x = 0, \qquad x \in \mathbb{R}, \qquad t > 0 \\ u(x,0) = \phi(x) \qquad x \in \mathbb{R}. \end{aligned}$$

To find the characteristics, we compute

$$\frac{d}{dt} u(x(t;\xi),t) = u_x \frac{dx}{dt} + u_t = u_x \left(\frac{dx}{dt} - t^2 \right).$$

The above is equal to zero when $dx/dt = t^2$, which implies $x(t;\xi) = t^3/3 + \xi$. Since $u(x(t;\xi),t) = u(x(0;\xi),0) = u(\xi,0) = \phi(\xi)$, because the solution is constant on characteristics, we find $u(x,t) = \phi(x - (1/3)t^3)$.

Question 7 [10 points]

Consider the PDE

$$u_t = u_x - u, \qquad x \in \mathbb{R}, \qquad t > 0$$
$$u(x,0) = \phi(x) \qquad x \in \mathbb{R},$$

where $u(x,t)$ decays to zero as $x \to \pm\infty$. Define the energy function $E(t) = \int_{-\infty}^{\infty} u^2(x,t)\mathrm{d}x$. Show that

$$E(t) \leq \int_{-\infty}^{\infty} \phi^2(x)\mathrm{d}x$$

for all $t \geq 0$.

We can compute

$$\frac{dE}{dt} = \int_{-\infty}^{\infty} 2uu_t\mathrm{d}x = \int_{-\infty}^{\infty} 2u(u_x - u)\mathrm{d}x = u^2(x,t)|_{-\infty}^{\infty} - 2\int_{-\infty}^{\infty} u^2(x,t)\mathrm{d}x = -2\int_{-\infty}^{\infty} u^2(x,t)\mathrm{d}x \leq 0,$$

where the boundary term is zero because $u(x,t)$ decays to zero as $x \to \pm\infty$. This implies that $E(t)$ is nonincreasing, so $E(t) \leq E(0) = \int_{-\infty}^{\infty} u^2(x,0)\mathrm{d}x = \int_{-\infty}^{\infty} \phi^2(x)\mathrm{d}x$.

Alternatively, one could solve the equation using the method of characteristics to find $u(x,t) = \phi(x+t)e^{-t}$. Next, notice

$$E(t) = \int_{-\infty}^{\infty} \phi^2(x+t)e^{-2t}\mathrm{d}x = e^{-2t}\int_{-\infty}^{\infty} \phi^2(z)\mathrm{d}z \leq \int_{-\infty}^{\infty} \phi^2(z)\mathrm{d}z = \int_{-\infty}^{\infty} \phi^2(x)\mathrm{d}x$$

using the change of variables $z = x + t$ and the fact that $e^{-2t} \leq 1$.

Question 8 [10 points]

Consider the Neumann problem

$$u_{xx} + u_{yy} + u_{zz} = f(x,y,z), \qquad (x,y,z) \in \Omega$$
$$\frac{\partial u}{\partial n} = 0 \qquad (x,y,z) \in \partial\Omega.$$

where Ω is some domain in $\mathbb{R}^3$. Show that in order for the PDE to have a solution it must be true that

$$\int_\Omega f(x,y,z)\ \mathrm{d}V = 0.$$

Hint: use the Divergence Theorem, which says

$$\int_\Omega \nabla \cdot \mathbf{F}(x,y,z)\ \mathrm{d}V = \int_{\partial\Omega} \mathbf{F} \cdot n\ \mathrm{d}S$$

for any vector field $\mathbf{F}$, where n is the outward pointing normal vector for Ω.

$$\begin{aligned}\int_\Omega f(x,y,z)\mathrm{d}V &= \int_\Omega \Delta u(x,y,z)\mathrm{d}V = \int_\Omega \nabla\cdot\nabla u(x,y,z)\mathrm{d}V = \int_{\partial\Omega} \nabla u(x,y,z)\cdot n\mathrm{d}S \\ &= \int_{\partial\Omega} \frac{\partial u}{\partial n}(x,y,z)\mathrm{d}S = \int_{\partial\Omega} 0\mathrm{d}S = 0,\end{aligned}$$

where we've applied the Divergence theorem to $\mathbf{F} = \nabla u$, used the fact that $\frac{\partial u}{\partial n} = \nabla u \cdot n$ by the definition of the normal derivative, and applied the boundary condition given above.

Question 9 [10 points]

Suppose u is a solution to Laplace's equation on the disk $D = \{0 \le r < 9\}$, with its value on the boundary being determined by $u(9,\theta) = 4 + \cos(2\theta)$ for $0 \le \theta < 2\pi$.

(i) What is the maximum value of u on $\{0 \le r \le 9\}$?
By the maximum principle, the maximum must occur on the boundary. Since the maximum value of cosine is 1, the maximum value of u is $4 + 1 = 5$.

(ii) Where does the maximum occur on $\{0 \le r \le 9\}$?
By the maximum principle, the maximum can only occur on the boundary (since the solution is not constant, because it is not constant on the boundary). Cosine achieves its maximum when $2\theta = 0, 2\pi$, so this occurs when $\theta = 0, \pi$. Thus, the maximum occurs at $(r,\theta) = (9,0)$ and $(9,\pi)$.

(iii) What is the value of u at the origin?
The value at the origin in the average value along the boundary, which gives:

$$u(0,0) = \frac{1}{2\pi}\int_0^{2\pi} [4 + \cos(2\theta)]\mathrm{d}\theta = 4.$$

Question 10 [10 points]

Find the steady state (or equilibrium) solution of

$$\begin{aligned} u_t = u_{xx} - u_x - 6u, &\qquad x \in \mathbb{R}, \qquad t > 0, \\ u(0,t) = 0, \qquad u(1,t) = b, &\qquad t \ge 0, \end{aligned}$$

where b is some fixed real constant.

By definition, the equilibrium solution is independent of time, so it solves $0 = u_{xx} - u_x - 6u$. The general solution to this equation is of the form $u(x) = Ae^{s_1 x} + Be^{s_2 x}$, where $s_{1,2}$ satisfy $s^2 - s - 6 = 0$. Therefore, the general solution is

$$u(x) = Ae^{3x} + Be^{-2x}.$$

The boundary condition at $x = 0$ implies $0 = u(0) = A + B$, so $B = -A$. Therefore, the boundary condition at $x = 1$ implies $b = u(1) = Ae^3 - Ae^{-2}$, and so $A = b/(e^3 - e^{-2})$. Therefore, the equilibrium solution is

$$u(x) = \frac{b}{e^3 - e^{-2}}(e^{3x} - e^{-2x}).$$