**Midterm – MA 561 – Fall 2016**

**Thursday, October 27, 2016**

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Name: ______________________________ BU ID: ______________________________

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Instructions: Please write clearly and show all work. If an answer is not justified, no points will be awarded. Points may be deducted for messy, unclear, or poorly explained work. Books, notes, and calculators are NOT permitted during this exam.

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Do not write in the following box.

| Problem | Possible | Score |
|---|---|---|
| 1 | 10 | |
| 2 | 10 | |
| 3 | 10 | |
| 4 | 10 | |
| 5 | 10 | |
| 6 | 10 | |
| 7 | 10 | |
| 8 | 10 | |
| 9 | 10 | |
| 10 | 10 | |
| Total | 100 | |

**Question 1 [10 points]** For each of the following PDEs

(i) $u_t - u_{xx} + x^2 = 0$

(ii) $u_x + e^x u_y = 0$

(iii) $u_x(1 + u_x^2)^{-1} + u_y(1 + u_y^2)^{-1} = 0$

answer the following questions:

(a) Is it linear or nonlinear?

(b) Is it homogeneous or inhomogeneous?

(c) What is its order?

Please be sure to give reasons for your answer.

**Question 2 [10 points]** Suppose both $u_1$ and $u_2$ are solutions of the PDE $\mathcal{L}u = g$, where $\mathcal{L}$ is some linear operator and $g$ is a given function. Show that $w = u_1 - u_2$ is a solution of $\mathcal{L}w = 0$.

**Question 3 [10 points]** Verify that $u(x, y) = f(x)g(y)$ is a solution of $uu_{xy} = u_x u_y$ for any given differentiable functions $f$ and $g$.

**Question 4 [10 points]**

Verify that the function

$$u(x,t) = \frac{x}{t^{3/2}} e^{-\frac{x^2}{4t}}$$

is a solution to the heat equation $u_t = u_{xx}$.

**Question 5 [10 points]** Consider the equation

$$2u_t + 3u_x = 0, \qquad x \in \mathbb{R}, \qquad t > 0$$
$$u(x,0) = \sin x \qquad x \in \mathbb{R}.$$

(i) Find an equation for the characteristics and sketch them in the $(x,t)$ plane.

(ii) Write down an explicit formula for the solution.

(iii) Sketch the solution for $t = 0$ and describe how it behaves for $t > 0$.

**Question 6 [10 points]** Solve the equation

$$
\begin{aligned}
u_t + t^2 u_x = 0, \qquad x \in \mathbb{R}, \qquad t > 0 \\
u(x,0) = \phi(x) \qquad x \in \mathbb{R}.
\end{aligned}
$$

**Question 7 [10 points]**

Consider the PDE

$$u_t = u_x - u, \qquad x \in \mathbb{R}, \qquad t > 0$$
$$u(x,0) = \phi(x) \qquad x \in \mathbb{R},$$

where $u(x,t)$ decays to zero as $x \to \pm\infty$. Define the energy function $E(t) = \int_{-\infty}^{\infty} u^2(x,t)\mathrm{d}x$. Show that

$$E(t) \leq \int_{-\infty}^{\infty} \phi^2(x)\mathrm{d}x$$

for all $t \geq 0$.

**Question 8 [10 points]**

Consider the Neumann problem

$$
\begin{aligned}
u_{xx} + u_{yy} + u_{zz} = f(x, y, z), \qquad & (x, y, z) \in \Omega \\
\frac{\partial u}{\partial n} = 0 \qquad & (x, y, z) \in \partial\Omega.
\end{aligned}
$$

where $\Omega$ is some domain in $\mathbb{R}^3$. Show that in order for the PDE to have a solution it must be true that

$$\int_\Omega f(x, y, z)\ \mathrm{d}V = 0.$$

Hint: use the Divergence Theorem, which says

$$\int_\Omega \nabla \cdot \mathbf{F}(x, y, z)\ \mathrm{d}V = \int_{\partial\Omega} \mathbf{F} \cdot n\ \mathrm{d}S$$

for any vector field $\mathbf{F}$, where $n$ is the outward pointing normal vector for $\Omega$.

**Question 9 [10 points]**

Suppose $u$ is a solution to Laplace's equation on the disk $D = \{0 \leq r < 9\}$, with its value on the boundary being determined by $u(9, \theta) = 4 + \cos(2\theta)$ for $0 \leq \theta < 2\pi$.

(i) What is the maximum value of $u$ on $\{0 \leq r \leq 9\}$?

(ii) Where does the maximum occur on $\{0 \leq r \leq 9\}$?

(iii) What is the value of $u$ at the origin?

## Question 10 [10 points]

Find the steady state (or equilibrium) solution of

$$u_t = u_{xx} - u_x - 6u, \qquad x \in \mathbb{R}, \qquad t > 0,$$
$$u(0,t) = 0, \qquad u(1,t) = b, \qquad t \geq 0,$$

where $b$ is some fixed real constant.