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TOPICS IN STABILITY THEORY FOR PARTIAL DIFFERENTIAL EQUATIONS

by

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TOPICS IN STABILITY THEORY FOR PARTIAL DIFFERENTIAL EQUATIONS

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Abstract

Determining the stability of solutions is central to the analysis of partial differential equation (PDE) models arising in applications, as it is typically the stable solutions that are observed in practice. Topics related to the stability analysis of parabolic PDEs are discussed, including techniques used in determining the linear, nonlinear, and global stability of stationary solutions. For linear stability, the focus is both on determining the behavior of solutions when the spectrum of the linear operator is known, but lacks a spectral gap, and on locating the spectrum. Regarding the former, four examples are analyzed using renormalization groups, scaling variables, and spectral decompositions. In this analysis, a novel technique is applied that separates the solution into two components that naturally reflect the advection properties of the linear operator, allowing for the application of scaling variables and the creation of a spectral gap. To address the latter, a model of bioremediation, a process for cleaning contaminated soil, is considered. In this example, locating the spectrum is less straightforward. Geometric singular perturbation theory is employed to construct a traveling wave solution, and its properties are subsequently used in locating the spectrum of the associated linearized operator, thus determining the spectral stability of the wave. Nonlinear stability is then discussed. In general, when the linear operator lacks a spectral gap, the effects of the nonlinearity are not well understood. However, detailed information can be obtained in specific examples, three of which are presented. Existing results for the heat equation with polynomial nonlinearity are reviewed, as well as new results for nonlinear PDEs in which the linear operator is that which arises in the stability analysis of the traveling front in Burgers equation. Using the technique introduced in the linear stability analysis, invariant manifolds are constructed in the phase space of perturbations of this front. As a result, the asymptotic form of solutions will be determined, illustrating why their algebraic temporal decay rate can be increased by working in appropriate algebraically weighted Banach spaces. Finally, global stability is discussed, including the development of a Lyapunov functional argument for the traveling front in Burgers equation.

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Chapter 1

Introduction

In this dissertation, I discuss several topics related to the stability analysis of parabolic partial differential equations (PDEs). In particular, techniques used in determining the linear, nonlinear, and global stability of stationary solutions will be considered. In addition to novel ideas, I include some well known results that I feel help to provide a more complete picture of the relevant concepts.

My purpose in organizing this dissertation as you find it is to provide both a brief introduction to stability theory and a discussion of some more advanced results. It is not intended to be an exhaustive account of the subject. Instead, I hope it may help connect some of the ideas found in introductory graduate texts on PDEs with current research in stability analysis.

Before beginning the discussion of PDEs, an overview of stability theory for ordinary differential equations (ODEs) will be given. Next, a brief introduction to some of the ideas that will be encountered in the study of PDEs will be presented. This chapter will conclude with an outline of the remainder of this dissertation.

1.1 A Brief Review of Stability Theory for Ordinary Differential Equations

In this section, we will briefly review some existing results regarding the stability of solutions to ODEs, so that we may subsequently compare them with the corresponding analysis for PDEs. Attention will be paid primarily to understanding concepts, rather than specific details. Further information may be found, for example, in [9], [28], [29], [30], [32], [45], and [50]. By studying the finite dimensional case first, some intuition can be gained that will be very useful in the study of the infinite dimensional case. As stated by Henry [31], "with a grounding in modern analysis, $\mathbb{R}^n$ looks much the same as any other Banach space."

Consider the following ordinary differential equation

$$w' = f(w), \tag{1.1}$$

where $w = w(t) \in \mathbb{R}^n$ and $f : \mathbb{R}^n \to \mathbb{R}^n$ is assumed to be C^r for $r \geq 1$, *i.e.* the function and its first r derivatives are assumed to be continuous. Under these assumptions, solutions to the above equation will exist at least locally in time. We are interested in studying the stability of a stationary solution, which in this finite dimensional case is simply an equilibrium point, $w(t) \equiv w_0$. In particular, we would like to determine under what conditions the solution is stable in the following sense.

Definition 1.1.1 *The solution $w(t) \equiv w_0$ of equation (1.1) is said to be* **stable** *if for any neighborhood V of w_0 there exists another neighborhood W such that any solution $w(t)$ with initial data in W is defined and lies in V for all $t > 0$. The solution w_0 is said to be* **asymptotically stable** *if it is stable and W can be chosen so that $w(t) \to w_0$ as $t \to \infty$. The solution w_0 is* **unstable** *if it is not stable.*

For a discussion of other types of stability, see the above mentioned references.

In order to determine if the solution $w(t) \equiv w_0$ is stable, we linearize the vector field around this solution. To do this, let $w(t) = w_0 + u(t)$, where it is understood that $u(t)$ is small. We then obtain

$$u' = Df(w_0)u + (f(w_0 + u) - Df(w_0)u) \equiv Au + N(u), \tag{1.2}$$

where $A = Df(w_0)$. Understanding the behavior of solutions to the above equation will provide a complete picture of the stability properties of the equilibrium solution. In particular, if $u \equiv 0$ is asymptotically stable, then so is $w \equiv w_0$. Thus, throughout this discussion we will focus on determining the stability of the zero solution.

1.1.1 Linear Equations

First we focus on the analysis of the linear part of the above vector field,

$$u' = Au. \tag{1.3}$$

In this case, solutions exist for all time and their behavior is completely determined by the spectrum of the matrix A, $\sigma(A)$, which is defined in the following way. Define the resolvent set of A as $\rho(A) = \{\lambda \in \mathbb{C} : (A - \lambda\mathbf{1}) \text{ is bijective }\}$. This implies that, for $\lambda \in \rho(A)$, $(A - \lambda\mathbf{1})^{-1}$ exists. The operator $(A - \lambda\mathbf{1})$ will fail to be bijective if and only if it has a nontrivial null space. The spectrum of A is then the complement of the resolvent set, $\sigma(A) = \mathbb{C} \setminus \rho(A)$. These are values of λ for which there exists a nonzero vector u satisfying $(A - \lambda\mathbf{1})u = 0$. Equivalently, they are the n (not necessarily distinct) zeros of the n-th order characteristic polynomial associated to the equation $Au = \lambda u$. We refer to elements of $\sigma(A)$ as eigenvalues and note that they are, in some cases, explicitly computable.

In addition, we associate to each eigenvalue an eigenvector or generalized eigenvector. If the algebraic multiplicity of the eigenvalue λ is k, $1 \le k \le n$, and its geometric multiplicity is l, $1 \le l \le k$, then there exist l nonzero eigenvectors v_m, $m = 1, \dots, l$ satisfying $(A - \lambda\mathbf{1})v_m = 0$, and $k - l$ nonzero generalized eigenvectors v_i, $i = 1, \dots, k - l$ satisfying $(A - \lambda\mathbf{1})v_i \neq 0$, $(A - \lambda\mathbf{1})^j v_i = 0$ for some $j = 2, \dots, k - l + 1$. The collection of eigenvectors and generalized eigenvectors associated to the matrix A defines a basis for $\mathbb{C}^n$.

Remark 1.1.2 *In the context of spectral analysis it is often convenient to work in $\mathbb{C}^n$,* *rather than $\mathbb{R}^n$, because complex eigenvalues naturally lead to complex eigenvectors. However, there is a well defined correspondence between the two, as the complex eigenvectors may be split up into their real and imaginary parts, thus defining a basis for the two dimensional subspace of $\mathbb{R}^n$ associated to the pair of complex conjugate eigenvalues.*

Define the following subsets of $\sigma(A)$:

$$\begin{aligned} \sigma_s(A) &= \{\lambda \in \sigma(A) \text{ such that } \mathrm{Re}(\lambda) < 0\} \\ \sigma_u(A) &= \{\lambda \in \sigma(A) \text{ such that } \mathrm{Re}(\lambda) > 0\} \\ \sigma_c(A) &= \{\lambda \in \sigma(A) \text{ such that } \mathrm{Re}(\lambda) = 0\}. \end{aligned} \tag{1.4}$$

Note that, because $\sigma(A)$ consists of a finite number of elements, these sets are well separated in the sense that the distance between them is a strictly positive number. In other words, there is a nonzero spectral gap between these sets: $\max \mathrm{Re}(\sigma_s) \leq -\delta_s < 0 < \delta_u \leq \min \mathrm{Re}(\sigma_u)$. In addition, let $\mathbb{E}^s$, $\mathbb{E}^u$, and $\mathbb{E}^c$ be the associated (generalized) eigenspaces. These three linear subspaces divide the phase space of the ODE into invariant sets. $\mathbb{E}^s$ and $\mathbb{E}^u$ may be characterized as containing only solutions that exponentially decay in forwards and backwards time, respectively. Solutions in the subspace $\mathbb{E}^c$ do not exhibit exponential growth or decay. Their size either remains bounded, as in the case of zero or purely imaginary eigenvalues whose geometric and algebraic multiplicities are equal, or may grow algebraically, as in the case when their algebraic multiplicity exceeds their geometric multiplicity.

One way to think about the behavior of solutions to equation (1.3) is to divide the possibilities into three cases (see figure 1·1):

1. The linear operator possesses at least one eigenvalue with positive real part. This necessarily implies instability, and there will exist solutions that grow exponentially fast as time increases.

2. The linear operator possesses only eigenvalues with negative real part. This necessarily implies asymptotic stability, and all solutions will decay exponentially fast as time increases.

3. The linear operator possesses eigenvalues with negative or zero real part. In this case one needs more information to determine stability, in particular the algebraic and geometric multiplicities of the neutral (zero real part) eigenvalues.

Regardless of which of the above characterizations is satisfied by $\sigma(A)$, the most important point is that, for linear systems, the behavior of solutions, and hence stability, may be completely determined.

An explicit formula for solutions to equation (1.3) may be obtained by exponentiating the matrix A:

$$u(t) = e^{tA}u_0, \tag{1.5}$$

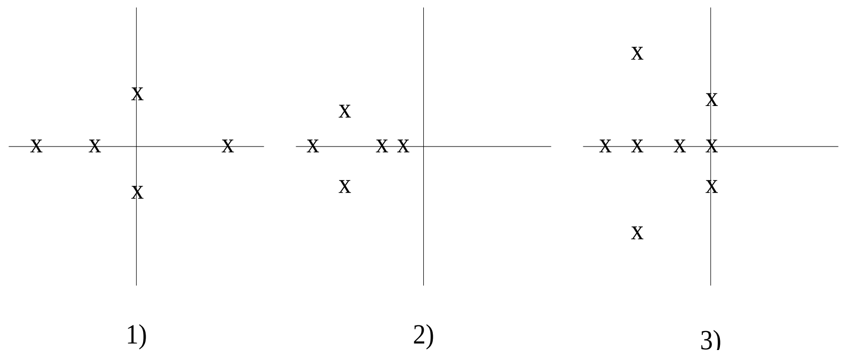

Figure 1·1: A schematic diagram of potential spectral scenarios in the complex plane for the finite dimensional case. 1) Eigenvalues exist in the open right half plane, indicating (exponential) instability. 2) All eigenvalues lie in the open left half plane, indicating (exponential) asymptotic stability. 3) Critical eigenvalues lie on the imaginary axis. Stability properties can be determined through further investigation.

where

$$e^{tA} = \sum_{n=0}^{\infty} \frac{t^n A^n}{n!}, \tag{1.6}$$

and $u_0 \in \mathbb{R}^n$ is the initial data. One can show that the object e^{tA}, also known as the semigroup associated to the matrix A, is well-defined as a bounded linear operator on $\mathbb{R}^n$, and that the function $u(t)$ as given in equation (1.5) does solve (1.3). Using the Jordan Normal Form of the matrix A and its associated semigroup we find the following. Denote each eigenvalue by $\lambda_j = a_j + ib_j$ for $j = 1, \ldots, n$ and a_j, $b_j \in \mathbb{R}$. Then each solution of equation (1.3) is a linear combination of functions of the form [32]

$$t^{k_j} e^{ta_j} \cos(b_j t), \quad t^{l_j} e^{ta_j} \sin(b_j t), \tag{1.7}$$

where k_j and l_j are less than or equal to the difference between the algebraic and geometric multiplicity of the eigenvalue λ_j. Using this explicit formulation of solutions for linear ODEs, the stability of zero solution can be completely determined.

1.1.2 Nonlinear Equations

We now return to the full nonlinear system, equation (1.2). First consider the case in which the matrix A is hyperbolic, *i.e.* $\sigma_c(A) = \varnothing$. In this case, the Hartman-Grobman Theorem states that, at least locally, the behavior of solutions to the full nonlinear system is topologically conjugate to the behavior of solutions to the linear system. In other words, for hyperbolic matrices the nonlinearity does not have a significant effect on the local stability of the zero solution.

When the matrix A is not hyperbolic, however, the situation is quite different. As a result of the Stable/Unstable/Center Manifold Theorem, we know that nonlinear analogues of $\mathbb{E}^s$, $\mathbb{E}^u$, and $\mathbb{E}^c$ exist locally, which are called the stable, unstable, and center manifolds, respectively. These manifolds are tangent at the equilibrium point to their respective linear subspaces and will be denoted by $\mathcal{W}^s$, $\mathcal{W}^u$, and $\mathcal{W}^c$. The stable and unstable manifolds are unique and of the same smoothness as the original vector field. The center manifold, however, is not necessarily unique and will not necessarily be analytic, even if the original vector field is.

This provides us with much information regarding the stability of the zero solution. In particular, $\sigma_u(A) \neq \varnothing$ implies (exponential) instability. On the other hand, $\sigma_u(A) = \sigma_c = \varnothing$ implies (exponential) stability. What happens if $\sigma_u(A) = \varnothing$ but $\sigma_c(A) \neq \varnothing$? This is a critical case as it was for the linear system. However, knowing the properties of A alone will not be sufficient for determining the stability properties. In this case, the nonlinearity must be taken into account. To do so, we must explicitly compute the flow on the center manifold. This can be done, at least in theory, by means of asymptotic expansions [9], and this procedure may allow one to determine the stability of the zero solution even in the presence of neutral eigenvalues.

In order to give a brief description of how such expansions can be computed, we state the center manifold theorem for ODEs. We will use projection operators to break the nonlinear problem into its stable and center components. First assume that the eigenvalues of the matrix A have either zero or negative real part, and that $\dim\mathbb{E}^s = j$ and $\dim\mathbb{E}^c = k$, where $j + k = n$. Using the (generalized) eigenvectors as a basis for each subspace, it is possible to define projection operators P_s and P_c associated to $\mathbb{E}^s$ and $\mathbb{E}^c$. Furthermore, we may define $A_s = A|_{\mathbb{E}^s}$ and $A_c = A|_{\mathbb{E}^c}$, where $\sigma(A_s) = \sigma_s(A)$ and $\sigma(A_c) = \sigma_c(A)$. We then have that

$$\begin{aligned} &e^{tA_s}P_s u \in \mathbb{E}^s, \quad ||e^{tA_s}P_s u|| \leq Ce^{-\delta t} \text{ for some } \delta > 0 \text{ and all } t \geq 0 \\ &e^{tA_c}P_c u \in \mathbb{E}^c, \quad ||e^{tA_c}P_c u|| \leq C_\epsilon e^{\epsilon t} \text{ for any } \epsilon > 0 \text{ and all } t \geq 0 \end{aligned} \tag{1.8}$$

Also, we may write $u = u_s + u_c \in \mathbb{R}^j \times \mathbb{R}^k$, where $u_s \in \mathbb{E}^s$ and $u_c \in \mathbb{E}^c$. We may then write equation (1.2) as

$$\begin{aligned} u_s' &= A_s u_s + P_s N(u_s, u_c) \\ u_c' &= A_c u_c + P_c N(u_s, u_c). \end{aligned} \tag{1.9}$$

The center manifold theorem states

Theorem 1.1.3 *[9] There exists a center manifold for equation (1.9),*

$$\mathcal{W}^c = \{u_s = h(u_c), \text{ for } |u_c| \leq \gamma\}, \tag{1.10}$$

where $\gamma > 0$ *is some (typically small) number and the function* h *satisfies* $h(0) = 0$, $h'(0) = 0$, *and* $h \in C^{r-1}$. *In addition, the flow on the center manifold is governed by the equation*

$$u_c' = A_c u_c + P_c N(h(u_c), u_c). \tag{1.11}$$

If the zero solution of equation (1.11) is asymptotically stable (unstable), then so is the zero solution of equation (1.9).

In addition, one can determine the asymptotic expansion for the function $h(u_c)$. If we insert the expression $u_s = h(u_c)$ into the first equation of (1.9), we obtain

$$h'(u_c)\left[A_c u_c + P_c N(h(u_c), u_c)\right] = A_s h(u_c) + P_s N(h(u_c), u_c).$$

Assume that $h(u_c)$ has the expansion $h(u_c) = \alpha_2 u_c^2 + \alpha_3 u_c^3 + \ldots$. By inserting this expression into the above equation and equating terms of equal order in u_c, it is possible, in principle, to compute the expansion of h accurate to any order in u_c (by determining the coefficients α_i) and, thus, determine the stability of the zero solution.

Recall the three distinguished scenarios for linear ODEs. The corresponding cases for nonlinear ODEs are:

1. The linear operator possesses at least one eigenvalue with positive real part. This necessarily implies instability, and there will exist solutions that leave a neighborhood of the zero solution exponentially fast as time increases.

2. The linear operator possesses only eigenvalues with negative real part. This necessarily implies asymptotic stability, and all solutions originating in a sufficiently small neighborhood of zero will decay exponentially fast as time increases.

3. The linear operator possesses eigenvalues with negative or zero real part. In this case one must compute the dynamics on the center manifold, thus incorporating the effects of the nonlinearity into the behavior of solutions.

Thus, we see that the presence of a nonlinearity does not (locally) affect stability in the first two cases. In the third case, however, the nonlinearity plays a key role in the behavior of solutions. Nevertheless, even for nonlinear ODEs, the stability of solutions can, in theory, be explicitly determined.

1.1.3 Global Stability

All of the above nonlinear results are local, in the sense that they hold only for sufficiently small initial data (recall definition 1.1.1 for asymptotic stability). This results from the fact that, when u is small, the nonlinearity $N(u)$ will be small compared to the linear part Au. For large u, however, the nonlinearity will be large relative to the linear part. Thus, the above analysis, which depends mainly on properties of the linear operator A, will not hold.

One way to prove global stability is construct a Lyapunov function for equation (1.1). A Lyapunov function is a function satisfying the following properties:

$$\begin{aligned}
&\text{i) } V(u) : \mathbb{R}^n \to \mathbb{R}, \quad V \in C^1 \\
&\text{ii) } V(0) = 0, \quad V(u) > 0 \text{ if } u \neq 0 \\
&\text{iii) } \frac{d}{dt} V(u(t)) = \nabla V(u(t)) \cdot (Au(t) + N(u(t))) < 0 \text{ for all } t \text{ and } u(t) \neq 0.
\end{aligned} \tag{1.12}$$

If such a function exists, then the solution $u \equiv 0$ is globally asymptotically stable, even if the linear operator possesses neutral eigenvalues. Intuitively, the above properties indicate that solutions evolve on level surfaces of V with strictly decreasing value and, therefore, must be converging to the zero level set, given by $\{u = 0\}$. This is a very nice result. However, its utility is limited by the fact that finding such a function is often quite difficult.

1.2 Stability Theory for Partial Differential Equations

The purpose of this dissertation is to present, at least to some extent, analogous results for PDEs. In order to focus on the general concepts involved, in this section we will not be concerned with details, which are left for the subsequent chapters. As we will see below, much of the theory and intuition for the ODE case will carry over. However, there are important differences that allow for more complicated behavior, even in the linear case.

Consider the following PDE:

$$u_t = Au + N(u), \tag{1.13}$$

where $u = u(x,t) \in X$ for some Banach space X, $A : X \to X$ is a linear (though not necessarily bounded) operator, and $N : X \to X$ is a nonlinear operator on X. Both A and N will be assumed to be sufficiently nice for the purpose of this discussion, and precise assumptions that they satisfy will be given in subsequent chapters. The situation we are concerned with is when u represents a perturbation of a stationary solution of interest, and so we are interested in the stability properties of the solution $u(x,t) \equiv 0$.

The definition of stability for stationary solutions to PDEs is analogous to that for ODEs, definition 1.1.1. We note, however, that the neighborhood V in the definition of stability is now defined in terms of the norm with which the Banach space X is equipped. By changing this norm one can dramatically change the stability properties of a solution, as we will see in section 2.4.

1.2.1 Linear Equations

First consider the linear PDE associated to equation (1.13),

$$u_t = Au. \tag{1.14}$$

As in the ODE case, the spectrum of A plays an important role in the dynamics of equation (1.14). However, there are two key differences between the infinite dimensional case of PDEs and the finite dimensional case of ODEs. One is that the spectrum of a linear operator on an infinite dimensional space is much more complicated and can include an infinite number of elements. Second, it can be, in general, quite difficult even to determine the spectrum of a linear operator on an infinite dimensional space.

The spectrum of A is defined in the same manner as in finite dimensions. The resolvent set of A is $\rho(A) = \{\lambda \in \mathbb{C} : (A - \lambda \mathbf{1}) \text{ is bijective}\}$, and the spectrum is $\sigma(A) = \mathbb{C} \setminus \rho(A)$. By the closed graph theorem, if $(A - \lambda\mathbf{1})$ is bijective, then $(A - \lambda\mathbf{1})^{-1}$, known as the resolvent operator, is a bounded linear operator on X. Notice that now there is more than one way for $(A - \lambda\mathbf{1})$ to fail to have a bounded inverse [15], [37]. For example: $(A - \lambda\mathbf{1})$ can have a nontrivial null space, $(A - \lambda\mathbf{1})$ can have a range that is not closed in X, or $(A - \lambda\mathbf{1})$ can have a range that is not dense in X. The first possibility defines the eigenvalues of A, often referred to as the point spectrum, and corresponds to elements of the spectrum in the finite dimensional case. The second and third possibilities define elements of the spectrum that are unique to PDEs.

We'd like to analyze the behavior of solutions to equation (1.14) in a manner analogous to the ODE case. We can still divide the spectrum into three parts, $\sigma_s(A)$, $\sigma_u(A)$, and $\sigma_c(A)$, with negative, positive, and zero real part respectively. In general, it is still the case that $\sigma_u \neq \varnothing$ implies (exponential) instability, while $\sigma_u = \sigma_c = \varnothing$ and $\sup \mathrm{Re}\sigma_s \leq -\delta < 0$ imply (exponential) stability. The main difference between the ODE and PDE case results from the fact that, for the PDE case, it can happen that $\sup \mathrm{Re}\sigma_s(A) = 0$. In this case, we say that there is no spectral gap between the stable and center modes.

The reason that linear PDE operators can lack a spectral gap is that their spectrum typically contains an infinite number of elements. As a result, elements of the spectrum may accumulate on the imaginary axis. In addition, there can be a continuous component, by which we mean any part of the spectrum that is not discrete. For example, the spectrum may contain a continuous curve, or an entire region, in the complex plane.

In order to illustrate this issue consider, for example, the operator $A = \partial_x^2$, acting on the Banach space $X = L^2(\mathbb{R})$. It is well known (and we will see explicitly below in section 3.1.1) that $\sigma(A) = (-\infty, 0]$. Thus, if we were to try and decompose the spectrum as in the ODE case, we would have $\sigma_s = (-\infty, 0)$ and $\sigma_c = \{0\}$. Working by analogy to the ODE case, because there is no unstable component of the spectrum, we can not immediately conclude that the zero solution is (exponentially) unstable. In order to determine the stability of the zero solution, we would need to determine the dynamics that result from the center directions. This can not be explicitly determined in the PDE case - at least not using techniques analogous to the ODE case. The reason is that this operator lacks a spectral gap, and thus there is no way to separate those solutions that decay exponentially from those that do not. In other words, one cannot use the spectrum to decompose the phase space of the PDE into invariant sets which have well separated exponential growth and decay, as in equation (1.8). As a result, there are no general methods for determining stability in this case.

If we return to the three scenarios presented for ODEs, we can create the corresponding scenarios for PDEs. However, we must add an additional possibility.

1. There exist elements of the spectrum of the linear operator that have positive real part. This necessarily implies instability, and there will exist solutions that grow exponentially fast as time increases.

2. All elements of the spectrum of the linear operator have negative real part and the spectrum is bounded away from the imaginary axis. This necessarily implies asymptotic stability, and all solutions will decay exponentially fast as time increases (but see Assumption 2.3.3.)

3. The spectrum of the linear operator contains elements that have negative or zero real part, and there exists a nonzero spectral gap between σ_s and the imaginary axis. In this case one needs more information to determine stability, but standard techniques, such as spectral decompositions (the linear analogue of invariant manifolds), exist that one may use in making this determination.

4. The spectrum of the linear operator contains elements that have negative or zero real part, but there does not exist a nonzero spectral gap. In this case there is no general method to determine the behavior of solutions.

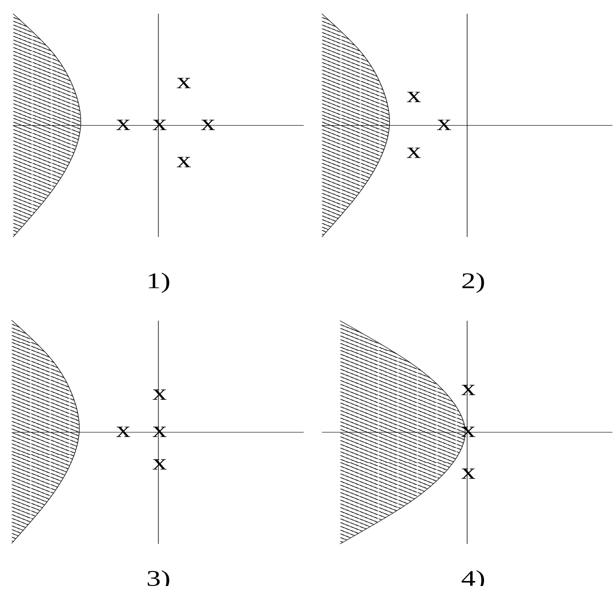

Figure 1·2: A schematic diagram of potential spectral scenarios in the complex plane for the infinite dimensional case. 1) Eigenvalues exist in the open right half plane, indicating (exponential) instability. 2) The entire spectrum lies in the open left half plane, indicating (exponential) asymptotic stability. 3) Critical eigenvalues lie on the imaginary axis with a nonzero spectral gap. Stability properties can be determined through further investigation, for example using spectral decompositions and center manifold theory. 4) Critical eigenvalues lie on the imaginary axis with no spectral gap. There is no general method to determine stability.

It is the presence of this fourth possibility that makes the study of stability in the context of PDEs much more complex, but also, in some sense, much more interesting.

Thus far we have assumed that $\sigma(A)$ is known. However, another issue that arises in the analysis of operators in linear PDEs is that computing the spectrum of the operator can often be quite difficult. To see why, recall that we are interested in the situation when the operator A is obtained by linearizing some PDE around a stationary solution of interest. For PDEs, stationary solutions will be independent of time but may still depend on the spatial variable. This will happen, for example, if we wish to study the stability of a stationary pulse or standing plane wave. As a result, the differential operator A may depend explicitly on the spatial variable, $A = A(x)$ say. An example of such an operator is $A(x) = \partial_x^2 + \alpha(x)\partial_x + \beta(x)$. Finding eigenvalues for this operator amounts to finding values of λ for which the nonautonomous ODE

$$\partial_x^2 u + \alpha(x)\partial_x u + \beta(x)u = \lambda u$$

has a solution $u \in X$. Because determining the behavior of solutions to nonautonomous ODEs can be quite difficult, so can finding eigenvalues of a spatially dependent linear operator. As a consequence, an extensive body of research has been developed to tackle just this issue. (See, for example, [51].) One particularly important tool used for locating eigenvalues of linear PDE operators is the Evans function, which will be discussed in chapter 4.

In the ODE case, we could solve a linear equation by exponentiating the matrix A. In the PDE case, defining the object e^{tA} is not always possible. However, for sufficiently nice operators A it can be done, and thus one can solve the linear PDE (1.14) by constructing the semigroup e^{tA}. The solution is then given by

$$u(t) = e^{tA}u_0, \tag{1.15}$$

where $u_0 \in X$ is the initial data. This formulation of the solution will be discussed in more detail below and, as in the ODE case, will be quite useful for determining the behavior of solutions.

1.2.2 Nonlinear Equations

Given that one can not in general determine the stability properties of the zero solution for linear PDEs, one can hardly expect to do so for nonlinear PDEs. However, in certain situations the results from the ODE case will carry over. In the case of ODEs, the Hartman-Grobman and Invariant Manifold theorems provide useful information about the effects of the nonlinearity on the behavior of solutions. There is no analogue of the Hartman-Grobman theorem for PDEs, but the invariant manifold theorems do have an analogue - as long as the linear operator possesses a spectral gap.

In general, the presence of unstable spectrum indicates (exponential) instability, and if all of the spectrum is contained in the left half of the complex plane and bounded away from the imaginary axis, then the zero solution is (exponentially) stable. If center directions exist (and there is a spectral gap between the center and stable directions), then a center manifold may be constructed and the dynamics within it computed, so that the stability of the zero solution may be determined.

In the absence of a spectral gap, there is no way, in general, to construct invariant manifolds and use them to determine stability. The reason is that in the ODE case, and also the PDE case, the spectral gap is used to obtain a separation between exponential growth and decay rates of solutions (see equation (1.8)), thus allowing one to construct contraction mappings whose fixed points are the desired invariant manifolds. In the absence of a spectral gap, these constructions fail, and one cannot necessarily conclude that stable, unstable, or center manifolds exist. In this case, the effects of the center directions, and the nonlinearity, are unclear.

1.2.3 Global Stability

All of the above mentioned results on nonlinear stability for PDEs are local in nature. If one wants to obtain global stability results, one must use different techniques that take into account non-local effects of the nonlinear terms. As with ODEs, Lyapunov functional arguments can be made, although additional technical complications arise. However, if one is able to find a Lyapunov functional, then the zero solution will be globally stable.

1.3 Summary of dissertation

The remainder of this dissertation will be divided into six chapters. The next chapter discusses preliminary ideas used in the study of the stability of solutions to linear PDEs. Basic definitions from spectral theory are given, as well as an example in which the spectrum of an unbounded linear operator is explicitly computed. Semigroups are then defined and used to formulate solutions to linear PDEs. Several examples are given, and the properties of semigroups are used to relate the spectral properties of the operator to the stability properties of the PDE. In addition, it is shown that the choice of the Banach space in which one works can have a nontrivial effect on the behavior of solutions. Finally, spectral decompositions, which divide the phase space of the PDE into invariant sets, are introduced.

The third chapter contains four concrete examples in which the spectrum of the linear operator can be explicitly determined, but lacks a spectral gap. For each example the spectrum of the linear operator will be computed using a Green's function. In addition, the stability and asymptotic (in time) form of solutions will be determined. This will be accomplished using several methods, including renormalization group maps and scaling variables. In particular, it will be shown that, by considering the PDE in an appropriately weighted space, one can affect not only the stability of the zero solution but also its asymptotic decay rate. In analyzing several of these examples, a novel technique will be applied that separates the solution into two components that naturally reflect the advection properties of the linear operator. One component will capture the far field behavior near $+\infty$, and the other will capture the far field behavior near $-\infty$. This allows for the application of scaling variables and the creation of a spectral gap. A spectral decomposition is then used to determine the asymptotic structure of solutions.

One additional example - a model of bioremediation, a process for cleaning contaminated soil - will be considered in the fourth chapter. The model involves a substrate (contaminant to be removed), electron acceptor (added nutrient), and microorganisms. Traveling wave solutions to the model exist and correspond to the motion of a biologically active zone, in which the microorganisms consume both substrate and acceptor. The goal is to prove spectral stability of the traveling wave, *i.e.* that the spectrum of the associated linear operator lies within the closed left half plane. In this example, locating the spectrum of the operator is much less straightforward than in the previous four examples. First, geometric singular perturbation theory will be employed to construct the traveling wave solution of the model. Second, properties of the wave that were elucidated by the construction will be used to locate the spectrum of the operator, thus determining the spectral stability of the wave.

The fifth chapter is a brief introduction to ideas used in the analysis of nonlinear PDEs. The existence and uniqueness of solutions will be discussed using the semigroup associated to the linear operator and the integral form of solutions. In addition, a center manifold theorem for PDEs will be given.

In general, in the absence of a spectral gap the effects of a nonlinear term are not well understood. However, detailed information can be obtained in specific examples, three of which will be presented in the sixth chapter. Existing results for the heat equation with polynomial nonlinearity will be discussed, using both renormalization group techniques and scaling variables. New results for nonlinear PDEs, in which the linear operator is that which arises in the study of the stability of the traveling front in Burgers equation, will also be given. The solution will be divided into three components: two far field and one near field. The far field components correspond to those used in the study of linear PDEs in the third chapter. The near field component is necessary for the study of the nonlinear equation, as it can be used to absorb any coupling between the components that results from the nonlinearity. Using this decomposition, scaling variables may then be applied, which allows for the application of center manifold theory and the determination of both the stability and asymptotic form of solutions. This technique will illustrate why the algebraic decay rate of solutions can be increased by considering the equation in appropriate algebraically weighted Banach spaces.

Finally, the last chapter will discuss the global stability of solutions. Lyapunov functionals for PDEs will be defined and the statement of LaSalle's Invariance principle will be given. An argument for the global stability of the traveling front in Burgers equation will then be presented. Using the Cole-Hopf transformation and results on linear Fokker-Planck equations, a Lyapunov functional will be constructed. It will then be indicated how LaSalle's invariance principle can be used to prove the global stability of the one-dimensional family of translates of the wave.

As mentioned above, included in this dissertation are various well known results that help place the problems studied into the larger context of stability analysis. The majority of the original research is contained in chapters 3.2-3.4, 4, 6.2, 6.3, and 7.2.

Chapter 2

Linear Partial Differential Equations: preliminary notions

In this chapter we begin our discussion of stability theory with linear PDEs. We introduce some preliminary ideas regarding spectral theory for unbounded operators, their associated semigroups, the existence and uniqueness of solutions, and spectral decompositions. This section is intended to be a brief introduction to these concepts. For more details, see the references herein.

2.1 Spectral Definitions

In this section, we give a short introduction to spectral theory that is focused on the ideas and definitions that will be used in the subsequent stability analysis. This material, including some proofs and further details, can also be found in [15], [31], [37], and [48].

Suppose we have a closed, linear operator

$$A : D(A) \subset X \to X, \tag{2.1}$$

where X is some Banach space.

Definition 2.1.1 *The* **resolvent set** *of A is defined to be*

$$\rho(A) = \{\lambda \in \mathbb{C} \mid (A - \lambda \mathbf{1}) : D(A) \to X \textit{ is bijective}\}. \tag{2.2}$$

The **spectrum** *of A is the complement of the resolvent set, $\sigma(A) = \mathbb{C} \setminus \rho(A)$.*

Due to the closed graph theorem, the resolvent set is equivalent to the set of complex numbers for which $(A - \lambda\mathbf{1})^{-1}$, known as the resolvent operator, is a bounded linear operator on X. The spectrum of an operator is therefore all complex numbers for which the resolvent fails to have a bounded inverse. We will focus on the following two disjoint subsets of the spectrum.

Definition 2.1.2 *A complex number λ is said to be an* **eigenvalue** *of A if there exists a nonzero element of the Banach space $v \in X$, known as the associated* **eigenfunction**, *such that $Av = \lambda v$. The (algebraic)* **multiplicity** *of an eigenvalue λ is the dimension of the set $\{v : (A - \lambda \mathbf{1})^k v = 0$ for some $k\}$. The* **point spectrum**, *$\sigma_{pt}(A)$, is then defined to be the set of all isolated eigenvalues of finite multiplicity. The* **essential spectrum**, *$\sigma_{ess}(A)$, is the remaining component of the spectrum, $\sigma_{ess}(A) = \sigma(A) \setminus \sigma_{pt}(A)$.*

In order to illustrate these ideas, consider the following linear operator,

$$Au = u_{xx} + \alpha u_x + \beta u, \tag{2.3}$$

where α and β are real numbers. We'd like to understand the spectrum of A in the space $X = L^2(\mathbb{R})$. To that end, consider the resolvent equation

$$(A - \lambda \mathbf{1})u = u_{xx} + \alpha u_x + \beta u - \lambda u = f. \tag{2.4}$$

If given any $f \in L^2$ we can find a $u \in L^2$ that satisfies the above equation, then $\lambda \in \rho(A)$. Taking the Fourier transform of the above equation, we see that

$$u(x) = \mathcal{F}^{-1}\left[\frac{\hat{f}(k)}{-k^2 + i\alpha k + \beta - \lambda}\right].$$

By the Fourier inversion theorem, if a function $\hat{w}$ is in L^2, then its inverse transform satisfies $w \in L^2$ with $||\hat{w}||_{L^2} = ||w||_{L^2}$. If $\lambda \notin \{-k^2 + i\alpha k + \beta \text{ for } k \in \mathbb{R}\}$, then we may bound

$$\begin{aligned}||\hat{u}(k)||^2_{L^2} &= \int |\frac{\hat{f}(k)}{-k^2 + i\alpha k + \beta - \lambda}|^2 dk \\ &\leq \sup_k \left(\frac{1}{|-k^2 + i\alpha k + \beta - \lambda|}\right)^2 \int |\hat{f}(k)|^2 dk \\ &\leq C(\lambda)||\hat{f}||^2_L = C(\lambda)||f||^2_{L^2}.\end{aligned}$$

Therefore, $||u||_{L^2} \leq C||f||_{L^2}$ and $\lambda \in \rho(A)$.

To see that $\lambda \in \{-k^2 + i\alpha k + \beta \text{ for } k \in \mathbb{R}\}$ implies $\lambda \in \sigma(A)$, we will use a proof by contradiction. We will construct a sequence of functions, u_n, such that $||u_n||_{L^2} = 1$ and $||(A-\lambda\mathbf{1})u_n||_{L^2} \to 0$. If we then define $f_n = (A-\lambda\mathbf{1})u_n$, we see that $||f_n||_{L^2} \to 0$. Suppose now that $\lambda \in \rho(A)$. Then $||(A - \lambda\mathbf{1})^{-1}|| = M$ for some M. We may then compute

$$||u_n||_{L^2} = ||(A - \lambda\mathbf{1})^{-1} f_n||_{L^2} \leq M||f_n||_{L^2}.$$

But, $||f_n||_{L^2} \to 0$, which is a contradiction, because $||u_n||_{L^2} = 1$ for all n. Thus, $\lambda \notin \rho(A)$.

To construct such a sequence u_n, consider the eigenvalue equation

$$u_{xx} + \alpha u_x + \beta u = \lambda u.$$

Solutions to this equation are given by

$$u_\lambda(x) = k_1 e^{s_+ x} + k_2 e^{s_- x}, \quad s_\pm = -\frac{\alpha}{2} \pm \frac{1}{2}\sqrt{\alpha^2 - 4(\beta - \lambda)}. \tag{2.5}$$

Regardless of the values of α and β, $u_\lambda \notin L^2$ for any value of λ. For $\lambda \in \{-k^2 + i\alpha k + \beta \text{ for } k \in \mathbb{R}\}$, k_1 and k_2 can be chosen so that u_λ is bounded. As a result, we can approximate the function u_λ by a sequence in L^2. For example, for a fixed λ define

$$u_\epsilon(x) = C(\epsilon) e^{-\epsilon x^2} u_\lambda(x),$$

where $C(\epsilon) = 1/||e^{-\epsilon x^2} u_\lambda(x)||_{L^2}$ so that $||u_\epsilon||_{L^2} = 1$. We may then compute

$$\begin{aligned}
||(A-&\lambda \mathbf{1})u_\epsilon||_{L^2}^2 = \\
&= C(\epsilon)^2 \int_{\mathbb{R}} |-4\epsilon x e^{-\epsilon x^2} u_\lambda'(x) + 4\epsilon^2 x^2 e^{-\epsilon x^2} u_\lambda(x) - 2\epsilon(1+\alpha x) e^{-\epsilon x^2} u_\lambda(x)|^2 dx \\
&\leq C(\epsilon)^2 \epsilon^2 \left(\int_{\mathbb{R}} |4x e^{-\epsilon x^2}|^2 \left(u_\lambda'(x)\right)^2 dx + \int_{\mathbb{R}} |(4\epsilon x^2 - 2(1+\alpha x)) e^{-\epsilon x^2}|^2 \left(u_\lambda(x)\right)^2 dx \right) \\
&\leq C \epsilon^{\frac{1}{2}} ||u_\lambda||_\infty^2,
\end{aligned}$$

which goes to zero as $\epsilon \to 0$. We can then let $u_n(x) = u_{\epsilon=1/n}(x)$, which completes the construction. Hence, $\sigma(A) = \{-k^2 + i\alpha k + \beta \text{ for } k \in \mathbb{R}\}$. Finally, we remark that because $u_\lambda \notin L^2$, there are no eigenvalues of A when considered as an operator on L^2. As a result, $\sigma(A) = \sigma_{ess}(A)$, and $\sigma_{pt}(A) = \varnothing$.

It is important to note that there are several definitions of the essential spectrum that are found in the literature. The above definition is found in [31]. Another frequently used definition can be found in [51], along with an explanation of its relation to the definition found above.

The reason we work with the point and essential spectrum as defined above, as opposed to other subsets of the spectrum found in the literature (for example the residual spectrum or approximate point spectrum), is as follows. In the context of stability analysis, one often works with linear operators that depend on the spatial variable, for example $A = A(x)$. Using the above definition of the essential spectrum, a result from [31] allows one to determine $\sigma_{ess}(A(x))$ by analyzing the asymptotic limits, $\lim_{x\to\pm\infty} A(x) = A^\pm$. This result can be quite useful in practice, and its precise statement is given by:

Theorem 2.1.3 *[31] Suppose that $m(x)$ and $n(x)$ are bounded real functions and that $\lim_{x\to\pm\infty} m(x)$, $n(x) = m^\pm$, $n^\pm$. In any of the spaces $L^p(\mathbb{R})$, $1 \leq p < \infty$, consider the* *following closed, densely defined linear operator:*

$$Au = \partial_x^2 u + m(x)\partial_x u + n(x)u.$$

If $S^{\pm} = \{\lambda : -k^2 + ikm^{\pm} + n^{\pm} - \lambda = 0,$ *for* $k \in \mathbb{R}\}$*, then* $S^{\pm}$ *are parabolas in the complex plane. In addition, define* $P \subset \mathbb{C}$ *such that* $\mathbb{C} \setminus P$ *is the component of* $\mathbb{C} \setminus (S^{+} \bigcup S^{-})$ *containing a right half-plane. Then the essential spectrum of* $A(x)$ *is contained in* P*, and in particular includes* $S^{+} \bigcup S^{-}$*. (See figure 2·1.)*

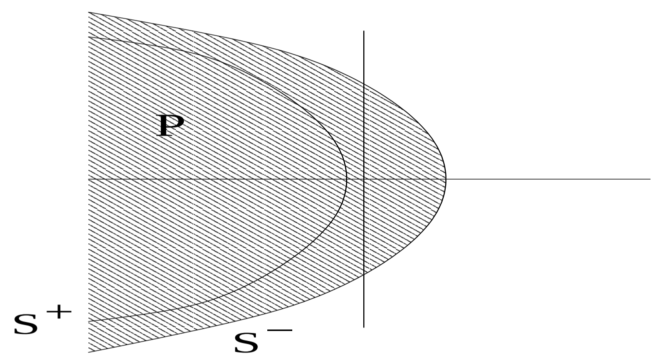

Figure 2·1: A schematic diagram of the parabolas $S^{\pm}$ and the region P, which contains the essential spectrum.

In order to see where the definition of the parabolas $S^{\pm}$ comes from, note that $\sigma(A^{\pm}) = \{\lambda : -k^2 + ikm^{\pm} + n^{\pm} - \lambda = 0, \text{ for } k \in \mathbb{R}\} = S^{\pm}$. It is a little more difficult to see how the spectrum of $A^{\pm}$ is related to $\sigma_{ess}(A)$. See [31] for details.

2.2 Semigroups

In this section we state results regarding the construction of the semigroup, $T(t) = e^{tA}$, associated to a linear operator A on a Banach space. Further details may be found in [15], [31], and [44]. The main questions we would like to address are: Under what conditions can a linear operator A be used to construct a one-parameter family of bounded linear operators, $\{T(t)\}_{t>0}$? What properties does the family $\{T(t)\}_{t>0}$ have, and how are they dependent upon the properties of A?

Definition 2.2.1 *A family* $\{T(t)\}_{t>0}$ *of bounded linear operators on a Banach space* X *is called a (one-parameter)* **semigroup** *on* X *if it satisfies*

$$\begin{aligned} T(t+s) &= T(t)T(s) \text{ for all } t,\ s > 0 \\ T(0) &= \mathbf{1}. \end{aligned} \tag{2.6}$$

To each semigroup we may associate a generator, A, which is defined in the following way.

Definition 2.2.2 *The linear operator A defined by*

$$D(A) = \{u \in X : \lim_{t\to 0^+} \frac{T(t)u - u}{t} \text{ exists}\}$$
$$Au = \lim_{t\to 0^+} \frac{T(t)u - u}{t} = \frac{dT(t)u}{dt}|_{t=0}, \text{ for } u \in D(A) \tag{2.7}$$

is the **infinitesimal generator** *of the semigroup $T(t)$, where $D(A)$ is the domain of A.*

One can check that the exponential of a matrix, $T(t) = e^{tA}$ given in equation (1.6), satisfies the above definition of a semigroup with the matrix A as its generator. We will frequently denote the semigroup generated by the operator A as $T(t) = e^{tA}$, just as in the finite dimensional case.

The idea is that a semigroup should be viewed as a linear dynamical system in which the parameter t represents time. The first condition given in equation (2.6), also known as the group property, states that flowing the system forward for a time $t + s$ is equivalent to flowing the system forward for a time t and then flowing forward for a time s. The second condition states that if no time has passed then the state of the system has not changed. Solutions are typically denoted by $u(t) = T(t)u_0 = e^{tA}u_0$.

As we know from the theory of ODEs, a dynamical system should evolve continuously in time (at least locally). Therefore, the semigroup $\{T(t)\}_{t>0}$ should be continuous in some sense with respect to the parameter t. The semigroup is a family of linear operators on a Banach space, and for such objects there are different types of continuity that result from placing different topologies (*e.g.* the uniform operator topology, strong operator topology, and weak operator topology) on the space of bounded linear operators, $\mathcal{L}(X)$. This leads to several classes of semigroups, three of which we will be relevant for this discussion.

Definition 2.2.3 *A one-parameter semigroup on a Banach space X is called* **uniformly continuous** *if for each $t > 0$ the map*

$$t \mapsto T(t) \in \mathcal{L}(X)$$

is continuous with respect to the uniform operator topology on $\mathcal{L}(X)$. In other words, for each $t > 0$, $\lim_{\epsilon\to 0^+} ||T(t+\epsilon) - T(t)||_{\mathcal{L}(X)} = 0$.

A uniformly continuous semigroup is quite nice, but it turns out that one can prove the object $\{T(t)\}_{t>0}$ will be uniformly continuous if and only if its generator A is bounded. Because most PDE operators are unbounded, we cannot expect them to generate uniformly continuous semigroups, and we must weaken the notion of continuity. We remark, however, that semigroups generated by finite dimensional matrices, *i.e.* those associated to ODEs, are uniformly continuous.

Definition 2.2.4 *A one-parameter semigroup on a Banach space X is called* **strongly continuous** *if for each $t > 0$, $u \in X$, the map*

$$t \mapsto T(t)u \in X$$

is continuous with respect to the strong operator topology on $\mathcal{L}(X)$. In other words, for each $t > 0$ and $u \in X$, $\lim_{\epsilon \to 0^+} ||T(t+\epsilon)u - T(t)u||_X = 0$.

It can be proven that the generators of strongly continuous semigroups are characterized by the following theorem, which is a slight generalization of the so-called Hille-Yosida theorem.

Theorem 2.2.5 *[15] A linear operator A is the (infinitesimal) generator of a strongly continuous semigroup satisfying $||T(t)|| \le Me^{\omega t}$ for $M \ge 1$ and $\omega \in \mathbb{R}$ if and only if*

1. *A is closed and densely defined.*
2. *For all λ satisfying $Re(\lambda) \ge \omega$, $\lambda \in \rho(A)$ and*

$$||((A - \lambda \mathbf{1})^{-1})^n|| \le \frac{M}{(Re(\lambda) - \omega)^n}, \tag{2.8}$$

for $n = 1, 2, 3, \ldots$.

For a given operator, the conditions stated in the above theorem are not necessarily easy to verify. We wish to emphasize, however, that in principle one can verify them based upon the spectral properties of A and its associated resolvent operator. In some cases, one can show directly (using definition 2.2.4) that an operator is the generator of a strongly continuous semigroup. We will encounter such an operator in chapter 3.1.2 below.

Many of the operators considered in the examples below will be the generators of semigroups that are not only strongly continuous, but also analytic.

Definition 2.2.6 *A family of operators $\{T(t)\}_{t>0}$ is called an* **analytic semigroup** *if*

1. *$\{T(t)\}_{t>0}$ is a strongly continuous semigroup on X.*
2. *The map $t \mapsto T(t)u$ is real analytic for each $t > 0$, $u \in X$.*

A characterization also exists for the generators of analytic semigroups. For any $0 < \phi < \pi/2$ and $\omega \in \mathbb{R}$, define the sector $S_{\phi,\omega}$ to be (see figure 2·2)

$$S_{\phi,\omega} = \{\lambda \in \mathbb{C} : |\arg(\lambda + \omega)| < \pi - \phi,\ \lambda \ne \omega\}. \tag{2.9}$$

Definition 2.2.7 *A linear operator A on a Banach space is a* **sectorial operator** *if it is closed and densely defined such that, for some $\phi \in (0, \pi/2)$, $M \ge 1$, $\omega \in \mathbb{R}$, the sector $S_{\phi,\omega}$ satisfies $S_{\phi,\omega} \subset \rho(A)$ and*

$$||(A - \lambda \mathbf{1})^{-1}|| \le \frac{M}{|\lambda - \omega|} \tag{2.10}$$

for all $\lambda \in S_{\phi,\omega}$.

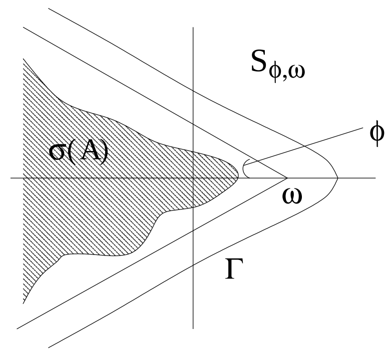

Figure 2·2: A schematic diagram of the sector $S_{\phi,\omega}$ and the curve Γ.

We then have the following theorem

Theorem 2.2.8 *[15] A linear operator A is the infinitesimal generator of an analytic semigroup satisfying $||T(t)|| \leq Me^{\omega t}$ if and only if it is a sectorial operator with sector $S_{\phi,\omega}$ for some $\phi \in (0, \pi/2)$.*

The real power of analytic semigroups results from the fact that they may be represented using contour integrals. If we let Γ be a contour in the complex plane as shown in figure 2·2, then it can be shown that the semigroup generated by a sectorial operator is given by

$$T(t) = e^{tA} = \frac{1}{2\pi i} \int_{\Gamma} (A - \lambda \mathbf{1})^{-1} e^{\lambda t} d\lambda. \tag{2.11}$$

Using this representation and properties of the resolvent operator, many important facts about analytic semigroups may be proved.

We now give an example of an operator that generates an analytic semigroup. Consider the following linear operator [31]

$$A = \partial_x^2, \quad D(A) = W^{2,p}(\mathbb{R}) \bigcap W_0^{1,p}(\mathbb{R}) \subset L^p(\mathbb{R}), \tag{2.12}$$

for $1 \leq p < \infty$. In order to show that A is the generator of an analytic semigroup, we must show that A is a sectorial operator. The greatest difficulty is proving estimate (2.10), which is a bound on the resolvent operator.

Using the result on the spectrum of the operator given in equation (2.3), we have that $\sigma(A) = (-\infty, 0]$. Given any $f \in L^p$, we must derive a bound for $w = (A - \lambda\mathbf{1})^{-1} f$. In other words, w solves the ODE

$$(\partial_x^2 - \lambda \mathbf{1})w = f.$$

Taking Fourier transforms, we see that

$$w(x) = \mathcal{F}^{-1}\left[\frac{\hat{f}}{-k^2-\lambda}\right] = \mathcal{F}^{-1}\left[\frac{1}{-k^2-\lambda}\right] * f(x).$$

Define $\Gamma_\lambda(x)$ to be the function satisfying $\hat{\Gamma}_\lambda(k) = 1/(-k^2-\lambda)$. If we let

$$G(x) = \frac{1}{\sqrt{4\pi}} \int_0^\infty t^{-\frac{1}{2}} e^{-t} e^{-\frac{x^2}{4t}}\, dt,$$

then $\hat{G}(k) = 1/(1+k^2) = -\hat{\Gamma}_1(k)$. Using the fact that

$$\hat{\Gamma}_\lambda(k) = \frac{\frac{1}{\lambda}}{-1-\frac{k^2}{\lambda}},$$

we may compute

$$\begin{aligned}
\Gamma_\lambda(x) &= \frac{1}{2\pi}\int_{\mathbb{R}} e^{ikx}\frac{1}{-\lambda-k^2}dk \\
&= -\frac{1}{\lambda 2\pi}\int_{\mathbb{R}} e^{ikx}\frac{1}{-1-\left(\frac{k}{\sqrt{\lambda}}\right)^2}dk \\
&= \frac{1}{2\pi\sqrt{\lambda}}\int e^{ip(\sqrt{\lambda}x)}\frac{1}{1+p^2}dp \\
&= -\frac{1}{\sqrt{\lambda}}G(\sqrt{\lambda}x).
\end{aligned}$$

Since we may estimate $||w||_{L^p} = ||\Gamma_\lambda * f||_{L^p} \leq ||\Gamma_\lambda||_{L^1}||f||_{L^p}$, we must derive a bound on $||\Gamma_\lambda||_{L^1}$. We have

$$\begin{aligned}
||\Gamma_\lambda||_{L^1} &= \frac{1}{|\sqrt{\lambda}|}\int |G(\sqrt{\lambda}x)|dx \\
&= \frac{1}{|\sqrt{\lambda}|}\int \frac{1}{\sqrt{4\pi}}\int_0^\infty t^{-\frac{1}{2}}e^{-t}e^{-\frac{\lambda x^2}{4t}}\, dtdx \\
&= \frac{1}{|\sqrt{\lambda}|^2}\int \frac{1}{\sqrt{4\pi}}\int_0^\infty t^{-\frac{1}{2}}e^{-t}e^{-\frac{y^2}{4t}}\, dtdy.
\end{aligned}$$

The integrals in the last term on the right hand side of the above estimate are independent of λ and convergent as long as $\lambda \notin (-\infty, 0]$ (due to the branch cut in the change of variables).

In fact, by integrating first with respect to y, we obtain

$$||\Gamma_\lambda||_{L^1} \leq \frac{1}{|\sqrt{\lambda}|^2}\int_0^\infty e^{-t}dt = \frac{1}{|\sqrt{\lambda}|^2}.$$

Therefore, we have that

$$||w||_{L^p} = ||(A - \lambda\mathbf{1})^{-1}f||_{L^p} = ||\Gamma_\lambda * f||_{L^p} \leq \frac{1}{|\lambda|}||f||_{L^p},$$

which is exactly the estimate in the definition of sectorial with $\omega = 0$ and $M = 1$. Hence, the operator ∂_x^2 generates an analytic semigroup on the space $L^p(\mathbb{R})$, for $1 \leq p < \infty$.

As with the characterization for strongly continuous semigroups, the condition that A be a sectorial operator is not, in general, easy to verify. However, there is a nice theorem that allows us, in many cases, to conclude that an operator A is the generator of an analytic semigroup.

Definition 2.2.9 *[15] Let $A : D(A) \subset X \to X$ be a linear operator on the Banach space X. An operator $B : D(B) \subset X \to X$ is called A-**bounded** if $D(A) \subset D(B)$ and if there exist constants a and b such that*

$$||Bu|| \leq a||Au|| + b||u|| \tag{2.13}$$

*for all $u \in D(A)$. The A-**bound** of B is*

$$a_0 = \inf\{a \geq 0 : \exists b \in \mathbb{R}^+ \text{ such that equation (2.13) holds}\}. \tag{2.14}$$

Using this definition, we have the following theorem.

Theorem 2.2.10 *[15] Suppose the operator A with domain $D(A)$ is the generator of an analytic semigroup on a Banach space X. Then the operator $A + B$ with domain $D(A)$ is also the generator of an analytic semigroup if B is A-bounded with A-bound 0.*

The reason why this theorem is particularly nice is that we will frequently be interested in operators of the form $A(x) = \partial_x^2 + \alpha(x)\partial_x + \beta(x)$. It was shown above that the Laplacian, ∂_x^2, with domain $D(\partial_x^2) = W_0^{1,2}(\mathbb{R}) \bigcap W^{2,2}(\mathbb{R})$ generates an analytic semigroup on L^2 (in fact, one could work in L^p for any $1 \leq p < \infty$) [31]. If the functions $\alpha(x)$ and $\beta(x)$ are, for example, continuous and bounded, then the operator $\alpha(x)\partial_x + \beta(x)$ is ∂_x^2-bounded with ∂_x^2-bound 0 [15]. Thus, such an operator $A(x)$ will also be the generator of an analytic semigroup. We will use this fact throughout the examples below.

2.3 Existence and Uniqueness of Solutions

Although there are no general existence theorems for solutions to PDEs, there are many techniques for proving the existence and uniqueness of solutions for certain classes of PDEs, for example Galerkin approximations [49]. We will concentrate on using semigroup techniques to prove such results, as they are quite natural for the setting of stability analysis.

A standard reference that uses the theory of analytic semigroups and their associated fractional Banach spaces is [31], in which a detailed account of the resulting properties of associated linear and nonlinear PDEs is given. For linear operators that generate only strongly continuous semigroups, we refer to [15] and [44].

Assume that a linear operator A is the generator of a strongly continuous semigroup on a Banach space X, which we will denote by $T(t) = e^{tA}$. Consider the following linear PDE.

$$u_t = Au, \quad u(0) = u_0. \tag{2.15}$$

Using the properties of a strongly continuous semigroup and the definition of its generator, one may show that the function $u(t) = e^{tA}u_0$ is the unique solution to equation (2.15). By a solution, we mean that $u(t)$ is continuous for all $t \geq 0$, continuously differentiable with $u(t) \in D(A)$ for all $t > 0$, and that it satisfies equation (2.15). Using the properties of strongly continuous semigroups given in Theorem 2.2.5, we find that the solution exists for all time and satisfies

$$||u(t)||_X = ||e^{tA}u_0||_X \leq Me^{\omega t}||u_0||_X.$$

Note that, because an analytic semigroup is also strongly continuous, this result applies to sectorial operators, as well. We have the following theorem:

Theorem 2.3.1 *If A is the generator of a strongly continuous semigroup on the Banach space X and the initial data satisfies $u_0 \in X$, then a unique solution to equation (2.15) exists for all $t \geq 0$ and is given by $u(t) = e^{tA}u_0$.*

One consequence of using the semigroup formulation of solutions is that we may, in many cases, determine the stability of the zero solution using the spectral properties of A.

Definition 2.3.2 *The* **spectral bound** *of a closed operator A is*

$$s(A) = \sup\{Re(\lambda) : \lambda \in \sigma(A)\}. \tag{2.16}$$

The **growth bound** *of a strongly continuous semigroup $\{T(t)\}_{t>0}$ with generator A is defined to be*

$$\omega_0(A) = \inf\{w \in \mathbb{R} : \exists M_\omega \text{ such that } ||T(t)|| \leq M_\omega e^{\omega t} \text{ for all } t \geq 0\}. \tag{2.17}$$

Intuitively, one might think that $s(A) = \omega_0(A)$. Unfortunately, this is not necessarily the case if the Banach space X is infinite dimensional. It can be proven, however, that

$$-\infty \leq s(A) \leq \omega_0(A). \tag{2.18}$$

As a result, we cannot even conclude, in general, that $s(A) \leq -\delta < 0$ implies stability. In fact, one can construct counterexamples for which $s(A) < 0$ but $\omega_0(A) > 0$.

For example (following [11]), consider the equation

$$u_t = xu_x,\ u(0) = u_0. \tag{2.19}$$

This equation appears to be relatively simple, as it is just an advection equation with $A = x\partial_x$. However, its behavior is quite interesting. One can directly check that the solution is given by $u(x,t) = T(t)u_0 = u_0(e^t x)$. In addition, one can verify that the semigroup is strongly continuous on the space $H^1(1,\infty)$, equipped with norm $||u||^2_{H^1} = ||u||^2_{L^2} + ||u_x||^2_{L^2}$.

We first compute the spectrum of $A = x\partial_x$ in $H^1(1,\infty)$. The eigenvalue equation is

$$\lambda u = xu_x,$$

and the solution is $u(x) = Cx^\lambda$, which is an element of $H^1(1,\infty)$ as long as $\mathrm{Re}(\lambda) < -1/2$. Thus, $\{\mathrm{Re}(\lambda) < -1/2\} \subset \sigma(A)$. To see that this is, in fact, all of the spectrum, consider

$$(A - \lambda\mathbf{1})u = f,$$

for $f \in H(1,\infty)$. This equation can also be solved explicitly:

$$u(x) = -x^\lambda \int_x^\infty s^{-\lambda-1} f(s)ds.$$

For a given λ, if $u \in H^1(1,\infty)$, then $\lambda \in \rho(A)$. One can directly verify that $u \in H^1(1,\infty)$ if $\mathrm{Re}(\lambda) > -1/2$ in the following manner.

One can directly compute

$$||u(x,t)||^2_{L^2} = \int_1^\infty u_0^2(e^t x)dx = e^{-t}\int_{e^t}^\infty u_0(z)dz \leq e^{-t}||u_0||^2_{L^2}. \tag{2.20}$$

Therefore, when considered as an operator on L^2, we see that $\omega_0(A) \leq -\frac{1}{2}$, and so $\{\mathrm{Re}(\lambda) \leq -1/2\} = \sigma(A)$. Hence,

$$||(A - \lambda\mathbf{1})^{-1} f||_{L^2} \leq C||f||_{L^2}.$$

In order to show that $\{\mathrm{Re}(\lambda) \leq -1/2\} = \sigma(A)$ in the space $H^1(1,\infty)$, as well, we must show that

$$||(A - \lambda\mathbf{1})^{-1} f||_{H^1} \leq C||f||_{H^1}.$$

The H^1 norm of $u = (A - \lambda\mathbf{1})^{-1} f$ may be written, using the above formula, as

$$\begin{aligned} ||u||_{H^1}^2 = &\int_1^\infty x^{2\lambda} \left(\int_x^\infty s^{-\lambda-1} f(s) ds \right)^2 dx \\ &+ \int_1^\infty \left(-\lambda x^{\lambda-1} \int_x^\infty s^{-\lambda-1} f(s) ds + x^{-1} f(x) \right)^2 dx. \end{aligned}$$

If $\mathrm{Re}(\lambda) > -\frac{1}{2}$, then the first term is bounded by $C||f||_{L^2}$ by the above result in L^2. The second term can be bounded directly, using the fact that

$$\begin{aligned} &\int_1^\infty \lambda x^{2(\lambda-1)} \left(\int_x^\infty s^{-\lambda-1} f(s) ds \right) dx \\ &\qquad \leq \lambda^2 \int_1^\infty x^{2(\lambda-1)} \left(\int_x^\infty s^{-2(\lambda+1)} ds \right) \left(\int_x^\infty f^2(s) ds \right) dx \\ &\qquad \leq ||f||_{L^2} \int_1^\infty x^{-3} dx. \end{aligned}$$

The other term may be bounded in a similar manner. Therefore, in $H^1(1,\infty)$, $\sigma(A) = \{\mathrm{Re}(\lambda) < -1/2\}$ and $s(A) = -1/2$.

We now claim that $\omega_0(A) \geq 1/2$. To prove this, note that we can chose a function $u_0 \in H^1(1,\infty)$ such that its support satisfies $\mathrm{supp}(u_0) \subset [e^t, \infty)$ and $||\partial_x u_0||_{L^2} = 1$. Then compute

$$\begin{aligned} ||T(t)u_0||_{H^1}^2 \geq ||\partial_x(T(t)u_0)||_{L^2}^2 &= \int_1^\infty \left(e^t u_0'(e^t x) \right)^2 dx \\ &= e^t \int_{e^t}^\infty \left(u_0'(z) \right)^2 dz = e^t ||u_0'||_{L^2} = e^t. \end{aligned}$$

This calculation implies that $\omega_0(A) \geq 1/2$ and, thus, the zero solution is unstable despite the spectral stability of the operator.

It turns out that determining the relationship between the spectrum of the generator and the growth rate of the semigroup is a difficult problem to solve. Its resolution lies in the Spectral Mapping Theorems [15], which provide sufficient conditions for

$$\sigma(T(t)) \setminus \{0\} = e^{t\sigma(A)}. \tag{2.21}$$

These theorems are beyond the scope of this dissertation. However, unless otherwise noted, the semigroups generated by the linear operators considered below will always satisfy a spectral mapping theorem.

Assumption 2.3.3 *For all linear operators studied in this dissertation, equation (2.21) is satisfied.*

This means that, in the linear examples we study, $\sup \mathrm{Re}\sigma(A) > 0$ will always imply instability, while $\sup \mathrm{Re}\sigma(A) \leq -\delta < 0$ will always imply stability.

Remark 2.3.4 *Because $s(A) \leq \omega_0(A)$, if $\sup Re\sigma(A) > 0$ then the zero solution will be unstable. The real difficulty lies is proving that spectral stability implies stability.*

Using the properties of semigroups, we have solved the linear stability problem for operators that either possess unstable elements of the spectrum or only stable elements of the spectrum with a bounded, nonzero distance from the imaginary axis. How can we analyze linear operators with spectrum that is arbitrarily close to, or lies on, the imaginary axis? We will turn to this question below. First we make a brief, but important, diversion.

2.4 Choose the Function Space Wisely!

In this section we briefly describe two examples that demonstrate the importance of the choice of the Banach space X with regard to the behavior of solutions to PDEs. The first is due to Sattinger [52], and the second can be found in [11] and [15]. Consider first the linear PDE

$$u_t = u_{xx} + \alpha u_x + \beta u, \tag{2.22}$$

where α and β are real numbers. The linear operator in this equation was considered above in section 2.1. Using theorem 2.2.10 and the discussion that follows, we know that the linear operator $A = \partial_x^2 + \alpha\partial_x + \beta$ generates an analytic semigroup on $X = L^2(\mathbb{R})$ with domain $D(A) = H_0^1 \bigcap H^2$. Furthermore, as shown in section 2.1, $\sigma(A) = \{\lambda = -k^2 + i\alpha k + \beta \text{ for } k \in \mathbb{R}\}$. This set is a parabola in the complex plane that opens to the left and has vertex β. Therefore, if $\beta > 0$ the zero solution to equation (2.22) will be unstable.

Suppose instead that we consider equation (2.22) in the space

$$Y = \{u \in L^2 : \int_{\mathbb{R}} (e^{\gamma x} u(x))^2 \, dx < \infty\}, \tag{2.23}$$

for some real number γ. We remark that Y is known as an exponentially weighted space. Elements of this space, if $\gamma > 0$ for example, must have a minimal amount of exponential decay at $+\infty$, but can actually grow exponentially at $-\infty$. Working in the Banach space

Y is equivalent to defining $v(x,t) \equiv e^{\gamma x} u(x,t)$ and studying the evolution of v in L^2. We find that v satisfies the equation

$$v_t = v_{xx} + (\alpha - 2\gamma) v_x + (\gamma^2 - \alpha\gamma + \beta) v. \tag{2.24}$$

Using the same technique as above, we find that the spectrum is now given by the set $\{\lambda = -k^2 + (\alpha - 2\gamma) ik + (\gamma^2 - \alpha\gamma + \beta) \text{ for } k \in \mathbb{R}\}$. Therefore, if $\alpha^2 > 4\beta$ and we choose γ so that $\alpha - \sqrt{\alpha^2 - 4\beta} < 2\gamma < \alpha + \sqrt{\alpha^2 - 4\beta}$, *ie* so that $\gamma^2 - \alpha\gamma + \beta < 0$, then the resulting spectrum will lie entirely in the left half plane and be bounded away from the imaginary axis. Thus, if one works in an exponentially weighted space, the zero solution is stable.

Next suppose that we analyze the same operator that was considered in equation (2.19) in section 2.3 above, $A = x\partial_x$, but work in $X = L^2(1,\infty)$ (rather than $X = H^1(1,\infty)$). By a direct calculation,

$$||u(t)||^2_{L^2(1,\infty)} = \int_1^\infty u_0^2(e^t x) dx = e^{-t} \int_{e^t}^\infty u_0^2(z) dz \le e^{-t} ||u_0||^2_{L^2(1,\infty)}.$$

Therefore, in this space the zero solution is stable. As seen in section 2.3 above, it is unstable in $H^1(1,\infty)$.

To see intuitively why the choice of the Banach space affects stability, consider again equation (2.22) and assume $\alpha > 0$, $\alpha^2 > 4\beta$. This implies that, for γ chosen as above, $\gamma > 0$. As a result, the requirement that $e^{\gamma x} u(x) \in L^2$ means that u must decay exponentially as $x \to \infty$. This is relevant because $\alpha > 0$ means that one aspect of the flow is that data is transported to the left, toward $-\infty$, at a rate α. By requiring exponential decay of u at $+\infty$, this somehow ensures that there isn't "too much stuff" brought in by the advection. The condition $\alpha^2 > 4\beta$ means that the instability caused by the source term, βu, isn't too strong to be overcome by the effects of the advection term, αu_x.

2.5 Spectral Decomposition of the Phase Space

In this section, we briefly discuss how one can decompose the phase space of a linear PDE using the spectrum of the linear operator. We focus primarily on the case when the unstable portion of the spectrum is empty and there is a nonzero spectral gap between the neutral and decaying modes.

Proposition 2.5.1 *[15] Let $A : D(A) \subset X \to X$ be a closed operator such that its spectrum $\sigma(A)$ can be decomposed into the disjoint union of two closed subsets $\sigma_c(A)$ and $\sigma_s(A)$, i.e.*

$$\sigma(A) = \sigma_c(A) \bigcup \sigma_s(A).$$

If σ_c is compact, then there exists a spectral decomposition $X = X_c \oplus X_s$ for A in the following sense.

1. *The restriction* $A_c \equiv A|_{X_c}$ *is bounded on the Banach space* X_c.
2. $A = A_c \oplus A_s$.
3. $\sigma(A_c) = \sigma_c(A)$ *and* $\sigma(A_s) = \sigma_s(A)$.

Assuming that A is the generator of a strongly continuous semigroup, $\sup \mathrm{Re}\sigma_s(A) \leq -\delta < 0$, and $\mathrm{Re}\sigma_c(A) = 0$, this decomposition implies the following. We may write $u(t) = u_c(t) + u_s(t)$, where $u_c(t) \in X_c$ and $u_s(t) \in X_s$, and $||u_s(t)||_X \leq Ce^{-\delta t}$. Therefore, in order to determine stability we need only determine the behavior of $u_c(t)$. In the case when $\sigma_c(A)$ consists of a finite number of isolated eigenvalues of finite multiplicity, making this determination is often relatively straightforward, in particular because X_c is finite dimensional. This allows one to compute the flow that results from the neutral eigenvalues, thus determining stability. An example of such a calculation will be given in section 3.1.2 below.

Although one is frequently interested in decomposing the phase space into the stable and center components, one can also decompose the phase space using arbitrary spectral sets, σ_1 and σ_2, as long as both are closed and disjoint in the extended complex plane, $\mathbb{C} \bigcup \{\infty\}$. For example, we could take σ_1 to be all eigenvalues whose real parts are greater then some fixed, negative number and $\sigma_2 = \sigma \setminus \sigma_1$.

Finally, we remark that the spectral decomposition of a linear operator is essentially an invariant manifold decomposition. In this case, the invariant manifolds are given by the linear subspaces X_c and X_s.

Chapter 3

Examples: location of spectrum known, no spectral gap

In this chapter, we consider four examples that illustrate a variety of techniques that can be used when studying the stability of PDEs, particularly in the absence of a spectral gap. In each of these examples we will be able to explicitly compute the spectrum of the linear operator and to determine the asymptotic (in time) behavior of solutions. These examples demonstrate that a wide range of behaviors can be displayed by linear operators with similar spectral pictures, even in the absence of nonlinear terms. In addition, they provide intuition that will be used in the analysis of nonlinear problems in chapter 6.

As we will see below, in all four of the examples the linear operator has a critical spectrum: a spectrum that lies in the left half of the complex plane but touches the imaginary axis with zero spectral gap. The reason such examples have been chosen is as follows. As discussed in section 2.2, if the linear operator has any spectrum in the right half plane, this indicates instability. If its entire spectrum is contained in the left half plane and has a nonzero distance from the imaginary axis, this indicates stability. If the operator possesses a finite number of eigenvalues on the imaginary axis and the rest of the spectrum lies in the left half plane with nonzero distance to the imaginary axis, then center manifold techniques that are quite similar to those in the ODE case may be used to determine stability. Thus, in some sense the most interesting case is when the entire spectrum of the linear operator lies in the closed left half plane but does not possess a spectral gap between its neutral (zero real part) and decaying (negative real part) modes. This scenario is unique to PDEs because it requires a spectrum with an infinite number of elements. As a result, we focus in this chapter on linear PDEs with this property.

3.1 Example 1: $u_t = u_{xx}$, the Heat Equation

We begin with the heat equation,

$$u_t = u_{xx}, \quad u(x,0) = u_0(x), \quad x \in \mathbb{R},\, t > 0, \tag{3.1}$$

which has been extensively studied. All the results in this section are well known, but it is useful to begin our investigation of specific ideas in stability analysis using such an example.

The operator $A = \partial_x^2$ with domain $D(\partial_x^2) = H_0^1 \bigcap H^2$ is the generator of an analytic semigroup on L^2, as shown in section 2.2 and [15], [31]. Therefore, given any initial data in L^2 solutions will exist for all $t > 0$. We note that one could also work in the space L^p,

$1 \leq p < \infty$, as well.

3.1.1 Spectral Properties

We compute the spectrum of the operator $A = \partial_x^2$ in the Banach space L^2. In order to determine $\rho(\partial_x^2)$, consider the equation

$$(\partial_x^2 - \lambda \mathbf{1})f = g. \tag{3.2}$$

By definition, $\lambda \in \rho(\partial_x^2)$ if given $g \in L^2$ we can find an $f \in L^2$ satisfying the above equation. Taking the Fourier transform of the above equation, we see that

$$\hat{f} = \frac{\hat{g}}{-k^2 - \lambda}.$$

Thus, we may find such an f if the inverse Fourier transform of the right hand side of the above equation lies in L^2. This will be the case exactly when $\lambda \neq -k^2$, for any $k \in \mathbb{R}$ (see the discussion following equation (2.4) in chapter 2.1).

An alternate way to determine the spectrum is to compute the Green's function associated to equation (3.2). This is a function satisfying

$$(\partial_x^2 - \lambda \mathbf{1})G(x, y) = \delta(x - y), \tag{3.3}$$

where $\delta(x - y)$ is the Dirac delta function. The solution to equation (3.2) may then be written

$$f(x) = \int G(x, y)g(y)dy \tag{3.4}$$

The Green's function for equation (3.2) is given by

$$G_\lambda(x, y) = -\frac{1}{2\sqrt{\lambda}}\left[H(y - x)e^{-\sqrt{\lambda}(y-x)} + H(x - y)e^{-\sqrt{\lambda}(x-y)}\right], \tag{3.5}$$

where the branch cut of $\sqrt{\ }$ is taken along the ray $(-\infty, 0]$. For any $\lambda \notin (-\infty, 0]$, $\sqrt{\lambda}$ will have positive real part. As a result, the Heaviside functions in equation (3.5) will "turn on" the exponentials exactly when they are exponentially decaying. Using this information about G_λ, we may estimate $||f||_{L^2} = ||G_\lambda * g||_{L^2} \leq ||G_\lambda||_{L^1}||g||_{L^2}$. Thus, $f(x)$ defined via equation (3.4) will lie in L^2 exactly when $\lambda \notin (-\infty, 0]$.

Therefore, we have found

$$\begin{aligned}\rho(\partial_x^2) &= \mathbb{C} \setminus \{(-\infty, 0]\} \\ \sigma(\partial_x^2) &= \{(-\infty, 0]\}\end{aligned} \tag{3.6}$$

Now that we have computed the spectrum of ∂_x^2, we will divide it into the point and essential spectrum. Consider the eigenvalue equation

$$\lambda f = \partial_x^2 f.$$

Solutions to this equation are given by $f(x) = e^{\pm\sqrt{\lambda}x}$, which are bounded only when $\lambda \leq 0$, but are not in L^2 for any value of λ. This shows that $\sigma(\partial_x^2) = \sigma_{ess}(\partial_x^2)$.

3.1.2 Behavior of Solutions

Because we are interested in stability, we would like to determine whether or not solutions to equation (3.1) decay to zero as $t \to \infty$. One can see that this is, in fact, the case, by solving the heat equation explicitly. One way to do this is to take the Fourier transform of equation (3.1) to obtain

$$\hat{u}_t = -k^2 \hat{u}. \tag{3.7}$$

Solving this ODE and taking the inverse Fourier transform, we find that the solution to equation (3.1) is

$$u(x,t) = \frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}} e^{-\frac{(x-y)^2}{4t}} u_0(y)dy. \tag{3.8}$$

The kernel of the above integral,

$$G(x,t) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}}, \tag{3.9}$$

is known as the fundamental solution of the heat equation. In addition, using equation (3.8), a direct calculation shows that

$$||u(t)||_{L^p} = ||G(t) * u_0||_{L^p} \leq ||G(t)||_{L^p} ||u_0||_{L^1} \leq \frac{C_p}{t^{\frac{1}{2}-\frac{1}{2p}}} ||u_0||_{L^1}. \tag{3.10}$$

This is already a nice result, but it is not easy to generalize because it is not often the case that one can explicitly solve a given PDE. Therefore, we would like to understand the behavior of solutions to (3.1) using the spectral properties of the operator.

Because the spectrum contains both neutral and stable eigenvalues with no spectral gap, there is no immediate way to determine stability. In particular, we cannot compute a center manifold associated to equation (3.1). It turns out, however, that solutions to the heat equation are scale invariant, meaning that the value of the solution at a time $t_1 > t_0$ may be found by appropriately scaling both x and the solution itself at time t_0 [6]. This can be seen directly by using equation (3.8). In addition, the fundamental solution to the heat equation, given in equation (3.9), satisfies this definition of scale invariance. This property may be exploited using, for example, both renormalization group methods and scaling variables in order to determine the stability of the zero solution. We will briefly outline both of these methods, as they will also be used in subsequent examples.

A detailed explanation of the method of renormalization groups in the context of PDEs can be found in [5] and [6]. The main ideas used in analyzing the heat equation are as follows. We seek to define a map, known at the renormalization group (RG) map, which possesses a stable fixed point: the scale invariant solution of the heat equation. Convergence to the scale invariant solution may then be proven by showing that the large time behavior of solutions can be found by iterating the RG map.

For notational convenience, take the initial time to be $t = 1$ and let the initial data be given by $u(x, 1) = u_0(x)$. Fix a number $L > 1$ and define

$$\begin{aligned} u_L(x,t) &= Lu(Lx, L^2t) \\ (R_L u_0)(x) &= u_L(x, 1), \end{aligned} \tag{3.11}$$

where $u(Lx, L^2t)$ is the solution to the heat equation. Thus, to iterate this map we solve the heat equation up to a finite time L^2, rescale x and the solution itself, and take the result to be the new initial data. We remark that the definition of this map, in particular the choice of powers of the constant L, is motivated by the form of the scale invariant fundamental solution to the heat equation, given in equation (3.9).

Using the Fourier transform, we may explicitly determine the form of the RG map. Solving equation (3.7) up to time L^2, we find that

$$\widehat{R_L u_0}(k) = e^{-k^2(1-L^{-2})}\hat{u}_0(L^{-1}k),$$

which has a line of fixed points given by constant multiples of

$$\hat{u}^*(k) \equiv e^{-k^2}. \tag{3.12}$$

This corresponds exactly to constant multiples of the scale invariant fundamental solution for $t = 1$. We denote by u^* its inverse Fourier transform.

We must now show that this line of fixed points is stable. Suppose $\hat{u}_0(0) = B_0$. We claim that $R_L^n u_0 \to B_0 u^*$ in an appropriately defined Banach space (see remark 3.1.1). Let $u_0 = B_0 u^* + g$, so that $\hat{g}(0) = 0$. Define the Banach space

$$Y = \{u : ||u|| = \sup(1+k^4)(|\hat{u}(k)| + |\hat{u}'(k)|) < \infty\}.$$

Using the fact that if $\hat{g} \in C^1$ then $|\hat{g}(\frac{k}{L})| \le \frac{|k|}{L}|\hat{g}'(\frac{k}{L})|$, we may estimate

$$\begin{aligned}
||R_L g|| &= \sup(1+k^4)(|\widehat{R_L g}(k)| + |(\widehat{R_L g})'(k)|) \\
&= \sup(1+k^4)[|e^{-k^2(1-L^{-2})}\hat{g}(L^{-1}k)| + |-2k(1-L^{-2})e^{-k^2(1-L^{-2})}\hat{g}(L^{-1}k) \\
&\quad + L^{-1}e^{-k^2(1-L^{-2})}\hat{g}'(L^{-1}k)|] \\
&\le \sup(1+k^4)\left[\left(\frac{|k|}{L} + 2\frac{|k|^2(1-L^{-2})}{L} + \frac{1}{L}\right)e^{-k^2(1-L^{-2})}|\hat{g}'(L^{-1}k)|\right] \\
&\le \frac{C}{L}\sup(1+k^4)|\hat{g}'(k)| \le \frac{C}{L}||g||.
\end{aligned} \tag{3.13}$$

Thus, R_L is a contraction and the line of fixed points $\{B_0\hat{u}^*\}$ is stable.

To see that iterates of the RG map correspond to the evolution of the heat equation, notice that $R_L^n = R_{L^n}$. This implies that

$$R_L^n u_0(x) = R_{L^n} u_0(x) = u_{L^n}(x,1) = L^n u(L^n x, L^{2n}).$$

If we now set $t = L^{2n}$, we find that the solution to the heat equation may be written

$$u(x,t) = L^{-n}(R_L^n u_0)(L^{-n}x) = t^{-\frac{1}{2}}(R_L^n u_0)(t^{-\frac{1}{2}}x). \tag{3.14}$$

Therefore, by iterating the RG map we can determine the long time behavior of solutions to the heat equation. This proves that, for initial data in the Banach space Y, the large time behavior of solutions to the heat equation is given by

$$u(x,t) \sim t^{-\frac{1}{2}} B_0 u^*(t^{-\frac{1}{2}}x) = \frac{\hat{u}_0(0)}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}}. \tag{3.15}$$

This result is particularly useful because it tells us not only that solutions will decay to zero, but it also tells us the asymptotic form of solutions as they decay, which can be nice in applications. In addition, as we will see in chapter 6, this method can be generalized to include nonlinear equations, as well. Although the zero solution is asymptotically stable, it is only algebraically stable. This makes intuitive sense - the lack of a spectral gap between the stable modes and the imaginary axis should prevent exponential decay of solutions.

Remark 3.1.1 *The choice of the Banach space Y may seem a bit strange at first. However,* *it is natural for obtaining the contraction estimate (3.13). This is an example of the fact that, not only can a Banach space be chosen in a way to ensure the stability of a solution, it can also be chosen in order to make certain estimates relatively straightforward.*

We now turn to the analysis of the heat equation using scaling variables, details of which may be found in [25] and [58]. The main idea is to use the scale invariance of the heat equation to find a suitable change of variables, known as scaling variables (or a similarity transformation), under which the dynamics of the equation are easier to analyze. For the heat equation, the dynamics are already quite easy to understand. One way in which the analysis could be simplified, for example, is if we changed coordinates so that the resulting linear operator possessed a spectral gap between its neutral and decaying modes. We will see that this is, in fact, possible using scaling variables.

Let $u(x,t)$ be the solution to the heat equation and define $w(\cdot,\cdot)$ by

$$\begin{aligned} u(x,t) &= \frac{1}{\sqrt{t+1}} w\left(\frac{x}{\sqrt{t+1}}, \log(t+1)\right) \\ \eta &= \frac{x}{\sqrt{t+1}}, \quad \tau = \log(t+1). \end{aligned} \tag{3.16}$$

We may then determine the equation of evolution for $w(\eta,\tau)$:

$$w_\tau = w_{\eta\eta} + \frac{1}{2}\eta w_\eta + \frac{1}{2} w, \tag{3.17}$$

where we denote the linear operator on the right hand side of the above equation by $\mathcal{L} = \partial_\eta^2 + \frac{1}{2}\eta\partial_\eta + \frac{1}{2}$.

Remark 3.1.2 *The reason for choosing the variables η and τ is as follows. The variable η results from the scale invariance of solutions to the heat equation. In particular, notice that the quantity $xt^{-\frac{1}{2}}$ appears in equation (3.14). The variable τ is chosen because a logarithmic scale will turn algebraic decay into exponential decay, thus allowing one to separate the rates of decay that result from elements of $\sigma(\partial_x^2)$ near $\lambda = 0$. The quantity $t+1$, rather than t, appears in their definitions so that, at the initial time $t = 0$, η and τ are well defined.*

This operator is the generator of a strongly continuous, but not analytic, semigroup on the algebraically weighted space

$$L^2(m) = \{u : (1+x^2)^{\frac{m}{2}} u \in L^2\}, \tag{3.18}$$

for $m \geq 0$. To see this, we obtain an explicit formula for the action of the semigroup by transforming equation (3.8) using the scaling variables [25]. We find

$$\begin{aligned} \widehat{e^{t\mathcal{L}} u_0}(k) &= e^{-a(\tau)k^2} \hat{u}_0(k e^{-\frac{\tau}{2}}) \\ e^{t\mathcal{L}} u_0(x) &= \frac{e^{\frac{\tau}{2}}}{\sqrt{4\pi a(\tau)}} \int e^{-\frac{(x-y)^2}{4a(\tau)}} u_0(y e^{\frac{\tau}{2}}) dy, \end{aligned} \tag{3.19}$$

where $a(\tau) = 1 - e^{-\tau}$. To show that the semigroup is strongly continuous, we must show that $||e^{t\mathcal{L}} u_0 - u_0||_{L^2(m)} \to 0$ as $t \to 0$ for each fixed $u_0 \in L^2(m)$. Because of the fact that a function satisfies $w \in L^2(m)$ if and only if $\hat{w} \in H^m$, this is equivalent to showing that $||\widehat{e^{t\mathcal{L}} u_0} - \hat{u}_0||_{H^m} \to 0$ as $t \to 0$.

We first prove that $||\widehat{e^{t\mathcal{L}} u_0} - \hat{u}_0||_{L^2} \to 0$ as $t \to 0$. Fix $\epsilon > 0$. We will show that there exists a $\tau_\epsilon > 0$ such that, for any $\tau \in (0, \tau_\epsilon)$, $||\widehat{e^{t\mathcal{L}} u_0} - \hat{u}_0||_{L^2}^2 \le \epsilon$. Because $\hat{u}_0 \in L^2$, there exists an L_ϵ such that $\int_{|k|>L_\epsilon} |\hat{u}_0(k)|^2 dk \le \epsilon/6$. Write

$$\begin{aligned} ||\widehat{e^{t\mathcal{L}} u_0} - \hat{u}_0||_{L^2}^2 = &\int_{|k|>L_\epsilon} |e^{-a(\tau)k^2} \hat{u}_0(k e^{-\frac{\tau}{2}}) - \hat{u}_0(k)|^2 dk \\ &+ \int_{|k|\le L_\epsilon} |e^{-a(\tau)k^2} \hat{u}_0(k e^{-\frac{\tau}{2}}) - \hat{u}_0(k)|^2 dk. \end{aligned}$$

The first term on the right hand side may be bounded as follows:

$$\begin{aligned} \int_{|k|>L_\epsilon} |e^{-a(\tau)k^2} \hat{u}_0(k e^{-\frac{\tau}{2}}) - \hat{u}_0(k)|^2 dk &= 2 \int_{|k|>L_\epsilon} e^{-2a(\tau)k^2} |\hat{u}_0(k e^{-\frac{\tau}{2}})|^2 + |\hat{u}_0(k)|^2 dk \\ &\le e^{\frac{\tau}{2}} \int_{|z e^{\frac{\tau}{2}}|>L_\epsilon} |\hat{u}_0(z)|^2 dz + \epsilon/6 \\ &\le \epsilon/3, \end{aligned}$$

where the last inequality follows by choosing $\tau_\epsilon < 1$, say, and L_ϵ larger if necessary so that $e^{\frac{1}{2}} \int_{e^{\frac{1}{2}}|z|>L_\epsilon} |\hat{u}_0(z)|^2 dz \le \epsilon/6$.

To bound the second term, we have

$$\int_{|k|\le L_\epsilon} |e^{-a(\tau)k^2}\hat{u}_0(ke^{-\frac{\tau}{2}}) - \hat{u}_0(k)|^2 dk \le 2\int_{|k|\le L_\epsilon} |e^{-a(\tau)k^2}\hat{u}_0(ke^{-\frac{\tau}{2}}) - e^{-a(\tau)k^2}\hat{u}_0(k)|^2 dk$$

$$+ 2\int_{|k|\le L_\epsilon} |e^{-a(\tau)k^2}\hat{u}_0(k) - \hat{u}_0(k)|^2 dk$$

$$\le \int_{|k|\le L_\epsilon} e^{-2a(\tau)k^2}\left(\hat{u}_0(ke^{-\frac{\tau}{2}}) - \hat{u}_0(k)\right)^2 dk + \int_{|k|\le L_\epsilon}\left(e^{-a(\tau)k^2} - 1\right)^2 |\hat{u}_0(k)|^2 dk$$

$$\le \int_{|k|\le L_\epsilon}\left(\hat{u}_0(ke^{-\frac{\tau}{2}}) - \hat{u}_0(k)\right)^2 dk + \sup_{|k|\le L_\epsilon}\left(e^{-a(\tau)k^2} - 1\right)^2 ||\hat{u}_0||^2_{L^2}$$

$$\le \epsilon/3 + \epsilon/3.$$

The last inequality follows by applying the dominated convergence theorem to the first integral and choosing τ_ϵ sufficiently small. One can show that $||\partial_k^m\left(\widehat{e^{t\mathcal{L}}u_0} - \hat{u}_0\right)||^2_{L^2} \to 0$ in a similar manner. Hence, $\mathcal{L}$ is the generator of a strongly continuous semigroup on $L^2(m)$.

Despite the lack of analyticity in the semigroup, the operator $\mathcal{L}$ is in many ways much easier to study than the operator ∂_x^2, due to the spectrum of $\mathcal{L}$.

Proposition 3.1.3 *[24], [25] Fix $m \ge 0$ and let $\mathcal{L}$ be the linear operator in $L^2(m)$, defined on its maximal domain. Then the spectrum of $\mathcal{L}$ is*

$$\sigma(\mathcal{L}) = \left\{\lambda \in \mathbb{C} : Re(\lambda) \le \frac{1-2m}{4}\right\}\bigcup\left\{-\frac{k}{2} : k = 0, 1, 2, \dots\right\}.$$

Moreover, if $m > \frac{1}{2}$ and if $k = 0, 1, 2, \dots$ satisfies $k + \frac{1}{2} < m$, then $\lambda_k = -\frac{k}{2}$ is an isolated eigenvalue with multiplicity 1. (See figure 3·1.)

In addition, the eigenfunctions are explicitly computable.

Proposition 3.1.4 *[24], [25] Fix $m > \frac{1}{2}$. Then for $k = 0, 1, 2, \dots,\ k + \frac{1}{2} < m$, the eigenfunctions ϕ_k associated to the eigenvalues λ_k are*

$$\phi_0(\eta) = \frac{1}{\sqrt{4\pi}}e^{-\frac{\eta^2}{4}}, \quad \phi_k(\eta) = \partial_\eta^k(\phi_0(\eta)). \tag{3.20}$$

We now apply the spectral decomposition described in section (2.5) with $m = 2$, $\sigma_c = \{0, -1/2\}$, and $\sigma_s = \sigma \setminus \sigma_c$. The space X_c is then two dimensional, with a basis given by the associated eigenfunctions, $\{\phi_0, \phi_1\}$. Furthermore, elements of X_s decay at a rate given by $\mathcal{O}(e^{-\frac{3}{4}\tau})$. We may then expand w as

$$w(\eta, \tau) = \alpha(\tau)\phi_0(\eta) + \beta(\tau)\phi_1(\eta) + \mathcal{O}(e^{-\frac{3}{4}\tau}). \tag{3.21}$$

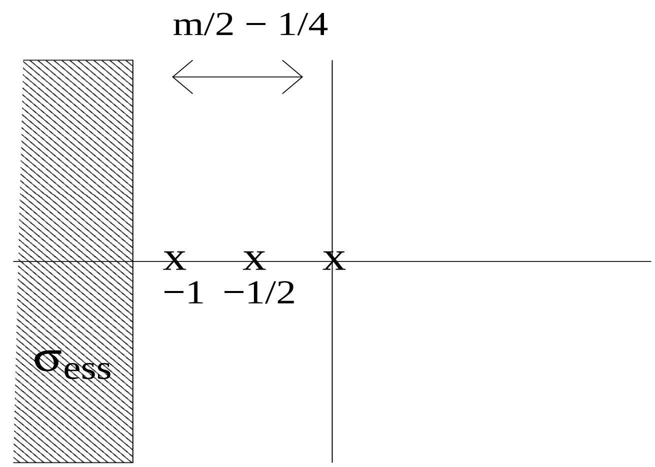

Figure 3·1: A schematic diagram of the spectrum of the operator $\mathcal{L}$, for $m = 3$.

In order to determine the functions $\alpha(\tau)$ and $\beta(\tau)$, plug the above expression into equation (3.17). To leading order,

$$\alpha_\tau \phi_0 + \beta_\tau \phi_1 = -\frac{1}{2}\beta(\tau)\phi_1.$$

Integrating the above equation from $\eta = -\infty$ to $\eta = +\infty$, we find that $\alpha_\tau = 0$, and so $\alpha(\tau) = \alpha_0$. Similarly, multiply the above equation by η and integrate to find $\beta_\tau = -\frac{1}{2}\beta$, which implies that $\beta(\tau) = e^{-\frac{1}{2}\tau}\beta_0$. Note that this is as expected, given σ_c. To determine the values of α_0 and β_0, notice that $\partial_\tau \int_{\mathbb{R}} w(\eta,\tau)d\eta = 0$ and $\partial_\tau \int_{\mathbb{R}} \eta w(\eta,\tau)d\eta = -\frac{1}{2}\int_{\mathbb{R}} \eta w(\eta,\tau)d\eta$. Therefore, we see that

$$\alpha(\tau) = \alpha_0 = \int_{\mathbb{R}} w(\eta, 0)d\eta, \quad \beta(\tau) = e^{-\frac{1}{2}\tau}\int_{\mathbb{R}} \eta w(\eta,0)d\eta. \tag{3.22}$$

If we translate this result back into the original (x, t) variables, we find

$$\begin{aligned} u(x,t) = &\left(\int_{\mathbb{R}} u(x,0)dx\right)\frac{1}{\sqrt{4\pi(t+1)}}e^{-\frac{x^2}{4(t+1)}} \\ &- \left(\int_{\mathbb{R}} xu(x,0)dx\right)\frac{1}{\sqrt{4\pi}(t+1)}\left(\frac{x}{2\sqrt{t+1}}\right)e^{-\frac{x^2}{4(t+1)}} + \mathcal{O}((t+1)^{-\frac{3}{4}}). \end{aligned} \tag{3.23}$$

Notice that the first term in equation (3.23) is exactly the leading order behavior that was obtained using the renormalization group map. The advantage to using scaling variables for this particular example is that, not only can one easily compute higher order terms in the expansion, but one can also see the underlying geometric structure that governs the behavior of solutions.

3.2 Example 2: $u_t = u_{xx} + c\tanh(\frac{c}{2}x)u_x$

This purpose of this example, along with the following two, is to illustrate the role that advection can play in the behavior of solutions to PDEs. We do not begin the discussion of advection with the equation

$$u_t = u_{xx} + ku_x, \tag{3.24}$$

because by changing to a moving coordinate frame, $\xi = x + kt$, the above equation can be transformed into the heat equation. Thus, solutions to equation (3.24) will exhibit essentially the same behavior as solutions to the heat equation. The only difference will be that data will be transported either to the left (if $k > 0$) or to the right (if $k < 0$) as it evolves.

Consider the equation

$$u_t = u_{xx} + c\tanh(\frac{c}{2}x)u_x, \tag{3.25}$$

and note that, using theorem 2.2.10, the linear operator in the above equation is the generator of an analytic semigroup in L^2. Thus, solutions exist for all $t \geq 0$. Because $\tanh(\frac{c}{2}x) \to \pm 1$ as $x \to \pm\infty$ exponentially fast, this equation is essentially equation (3.24) with $k = c$ for $x > 0$ and equation (3.24) with $k = -c$ for $x < 0$. As a result, information will be transported toward $x = 0$ on both the negative and positive half lines. This behavior is important, and we will refer to it again below.

Remark 3.2.1 *The study of this example was partially motivated by its relationship with Burgers equation. As will be discussed below, this equation is the integrated form of the equation studied in section 3.4, which is the equation one obtains after linearizing around the traveling front (with wavespeed c) in Burgers equation.*

3.2.1 Spectral Properties

We begin our investigation of the stability of the zero solution in equation (3.25) by determining the spectrum of the associated linear operator, $A_2 = \partial_x^2 + c\tanh(\frac{c}{2}x)\partial_x$, on $L^2(\mathbb{R})$. First note that theorem 2.1.3 implies that the boundary of the essential spectrum is given by $\{\lambda = -k^2 + ick : k \in \mathbb{R}\}$. This is a parabola in the complex plane that can be written $\{\mathrm{Re}(\lambda) = -(\mathrm{Im}(\lambda))^2/c^2\}$. (See figure 3·2a.) Next, we consider the equation

$$(A_2 - \lambda)u = u_{xx} + c\tanh(\frac{c}{2}x)u_x - \lambda u = 0. \tag{3.26}$$

Notice that under the transformation $u(x) = \text{sech}(\frac{c}{2}x)w(x)$, the above equation becomes

$$w_{xx} - (\frac{c^2}{4} + \lambda)w = 0.$$

This equation is explicitly solvable, and we find that the complete solution to (3.26) is given by

$$u_\lambda(x) = k_1 \text{sech}(\frac{c}{2}x)\exp[\sqrt{(\frac{c^2}{4} + \lambda)}x] + k_2 \text{sech}(\frac{c}{2}x)\exp[-\sqrt{(\frac{c^2}{4} + \lambda)}x], \quad (3.27)$$

where k_1 and k_2 are arbitrary constants. Due to the factor $\text{sech}(\frac{c}{2}x)$, the function $u_\lambda(x)$ will decay to zero exponentially as $x \to \pm\infty$ as long as $|\text{Re}\sqrt{\frac{c^2}{4} + \lambda}| < \frac{c}{2}$. This condition is satisfied exactly when λ lies to the left of the parabola defining the boundary of the essential spectrum. (See figure 3·2a.) Therefore, $\{\text{Re}(\lambda) \leq -(\text{Im}(\lambda))^2/c^2\} \subset \sigma(A_2)$.

In order to show that this is all of the spectrum, consider the equation

$$(A_2 - \lambda\mathbf{1})u = u_{xx} + c\tanh(\frac{c}{2}x)u_x - \lambda u = f. \quad (3.28)$$

As in the previous example, one can solve this equation using a Green's function, which is a function $G_\lambda(x, y)$ satisfying

$$(A_{2,x} - \lambda)G_\lambda = \delta(x - y), \quad (3.29)$$

where $A_{2,x}$ denotes the operator A_2 acting on the x variable. The solution is then given by

$$u(x) = \int_{\mathbb{R}} G_\lambda(x, y)f(y)dy. \quad (3.30)$$

A direct calculation shows that the Green's function is given by

$$\begin{aligned} G_\lambda(x, y) = &-\frac{1}{2\sqrt{(\frac{c^2}{4} + \lambda)}}\cosh(\frac{c}{2}y)\text{sech}(\frac{c}{2}x) \cdot \\ &\left[H(y - x)e^{-\sqrt{(\frac{c^2}{4}+\lambda)}(y-x)} + H(x - y)e^{-\sqrt{(\frac{c^2}{4}+\lambda)}(x-y)}\right]. \end{aligned} \quad (3.31)$$

In order to understand the behavior of the Green's function, let's determine its behavior as $|x|$, $|y| \to \infty$ for λ in different regions of the complex plane. First note that the parabola which determines the location of the essential spectrum is exactly the values of λ for which

$|\mathrm{Re}\sqrt{(\frac{c^2}{4}+\lambda)}| = \frac{c}{2}$. When λ is to the right of this parabola, the real part is larger than $c/2$, and for λ to the left of the parabola, the real part is less than $c/2$.

Next notice that, in the above formula, the factor $\cosh(\frac{c}{2}y)\mathrm{sech}(\frac{c}{2}x)$ will decay at infinity whenever $|x| > |y|$ with a rate given by $e^{-\frac{c}{2}|\xi|}$ (where $\xi = x \pm y$), and it will grow at infinity with a corresponding rate whenever $|y| > |x|$. Also, the Heaviside functions "turn on" the exponentials whenever their behavior is "nice". Thus, the exponential terms will never grow at a rate larger than $e^{\frac{c}{2}|\xi|}$, and for λ to the right of the parabola defining the essential spectrum, they will decay at least as fast as $e^{-(\frac{c}{2}+\delta)|\xi|}$, for some $\delta > 0$.

Thus, whenever λ is to the right of this parabola, the Green's function will decay at infinity regardless of the behavior of $\cosh(\frac{c}{2}y)\mathrm{sech}(\frac{c}{2}x)$, because of the fast decay of the exponentials. However, when λ is to the left of this parabola, the exponentials will be unable to compensate for any growth in the preceding factor, and so G_λ will grow exponentially at infinity. Exactly on the parabola, the exponential growth and decay balances, and so the Green's function will be bounded at infinity, but not necessarily decay.

This information regarding the Green's function implies that the spectrum of A_2 includes the parabola $\{\mathrm{Re}(\lambda) = -\frac{(\mathrm{Im}(\lambda))^2}{c^2}\}$ and its interior. The interior is filled with eigenvalues and the parabola is the boundary of the essential spectrum. We know that elements of the complex plane that lie to the right of the parabola are elements of the resolvent set, because the Green's function decays exponentially (in space) there. This implies that $u(x)$, given by equation (3.30), is in L^2. To see this rigorously, we may compute

$$\begin{aligned}
||u||_{L^2} &= ||\int_{\mathbb{R}} G_\lambda(x,y)f(y)dy||_{L^2} \\
&= \frac{1}{|2\sqrt{\frac{c^2}{4}+\lambda}|}||\int H(y-x)e^{\left(-\sqrt{\frac{c^2}{4}+\lambda}+\frac{c}{2}\right)(y-x)} e^{-\frac{c}{2}(y-x)}\cosh(\frac{c}{2}y)\mathrm{sech}(\frac{c}{2}x)f(y)dy \\
&\quad + \int H(x-y)e^{\left(-\sqrt{\frac{c^2}{4}+\lambda}+\frac{c}{2}(x-y)\right)} e^{-\frac{c}{2}(x-y)}\cosh(\frac{c}{2}y)\mathrm{sech}(\frac{c}{2}x)f(y)dy||_{L^2} \\
&\leq \frac{C}{|2\sqrt{\frac{c^2}{4}+\lambda}|}||\int [H(y-x)e^{\left(-\sqrt{\frac{c^2}{4}+\lambda}+\frac{c}{2}\right)(y-x)} \\
&\quad + H(x-y)e^{\left(-\sqrt{\frac{c^2}{4}+\lambda}+\frac{c}{2}\right)(x-y)}]f(y)dy||_{L^2} \\
&= \frac{C}{|2\sqrt{\frac{c^2}{4}+\lambda}|}||\tilde{G}_\lambda * f||_{L^2} \leq \frac{C}{|2\sqrt{\frac{c^2}{4}+\lambda}|}||\tilde{G}_\lambda||_{L^1}||f||_{L^2} \\
&\leq \frac{C}{|2\sqrt{\frac{c^2}{4}+\lambda}||\sqrt{\frac{c^2}{4}+\lambda}-\frac{c}{2}|}||f||_{L^2},
\end{aligned}$$

where

$$\tilde{G}_\lambda(z) = H(-z)e^{\left(-\sqrt{\frac{c^2}{4}+\lambda}+\frac{c}{2}\right)(-z)} + H(z)e^{\left(-\sqrt{\frac{c^2}{4}+\lambda}+\frac{c}{2}\right)(z)}.$$

Remark 3.2.2 *When $\lambda = -\frac{c^2}{4}$ the Green's function has a singularity, because that is the branch point of the function $\sqrt{(\frac{c^2}{4}+\lambda)}$. This fact is related to the maximum decay one can get when considering the operator in an exponentially weighted space. (See equation (3.34).)*

3.2.2 Behavior of Solutions

Using the same change of coordinates that was used to solve the eigenvalue problem, we may explicitly solve equation (3.25). The semigroup is given by

$$e^{tA_2}u_0(x) = \frac{1}{\sqrt{4\pi t}}\operatorname{sech}(\frac{c}{2}x)e^{-\frac{c^2}{4}t}\int_{\mathbb{R}} e^{-\frac{(x-y)^2}{4t}}\cosh(\frac{c}{2}y)u_0(y)dy. \tag{3.32}$$

At first glance it may seem that this formula implies that solutions will decay exponentially with a rate given by $e^{-\frac{c^2}{4}t}$. However, this is not the case. In fact, exponential decay is not expected given the lack of a spectral gap between the stable part of the spectrum and the imaginary axis. If one expands the hyperbolic cosine in the integrand in terms of exponentials and completes the square in each of the resulting terms, one will see that this exponential decay term cancels with a similarly growing exponential term. In particular, we find that

$$\begin{aligned} e^{tA_2}u_0(x) = &\frac{1}{\sqrt{4\pi t}(1+e^{-cx})}\int_{\mathbb{R}} e^{-\frac{(x+ct-y)^2}{4t}}u_0(y)dy \\ &+ \frac{1}{\sqrt{4\pi t}(1+e^{+cx})}\int_{\mathbb{R}} e^{-\frac{(x-ct-y)^2}{4t}}u_0(y)dy, \end{aligned} \tag{3.33}$$

and we note that the integration kernels are now exactly those of equation (3.24), with $k = \pm c$. The first term in the above equation is exponentially small for $x < 0$, and for $x > 0$ it is essentially advection to the left, toward $x = 0$. The second term is exponentially small for $x > 0$, and for $x < 0$ it is essentially advection to the right, toward $x = 0$. Thus, our initial understanding of the role of the advection term was fairly accurate.

We now determine the asymptotic form and temporal decay rate of solutions to equation (3.25). It was described in chapter 2.4 how one can sometimes affect the stability of the zero solution by appropriately choosing the Banach space X. We will see that one can do so for equation (3.44).

Because the spectrum of A_2 touches the imaginary axis at the origin, one might try to shift it to the left, as in section 2.4, by considering the equation in an exponentially weighted space. To that end, consider

$$Z = \{u \in L^2 : \cosh(\frac{c}{2}x)u \in L^2\}.$$

Studying equation (3.25) in the space Z is equivalent to studying the evolution of $v = \cosh(\frac{c}{2}x)u$ in L^2. We find that v satisfies

$$v_t = v_{xx} - \frac{c^2}{4}v.$$

Note that the spectrum of the linear operator is now given by $\{-k^2 - \frac{c^2}{4}$ for $k \in \mathbb{R}\}$. Thus, we expect that solutions will decay like $e^{-\frac{c^2}{4}t}$ as $t \to \infty$. To see that this is the case, notice that if we further define $w = e^{\frac{c^2}{4}t}v$, then w satisfies the heat equation. Thus, the techniques of the previous section tell us that

$$w(x,t) \sim \frac{\hat{w}(0)}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}}.$$

As a result, whenever the initial data satisfies $u_0 \in Z$, or $\cosh(\frac{c}{2}x)u_0 \in L^2$, the solution will satisfy

$$u(x,t) \sim \frac{\int \cosh(\frac{c}{2}y)u_0(y)dy}{\sqrt{4\pi t}} \operatorname{sech}(\frac{c}{2}x) e^{-\frac{c^2}{4}t} e^{-\frac{x^2}{4t}}. \tag{3.34}$$

The drawback with this method is that it requires that the initial data lie in Z, which is a fairly restrictive space. This guarantees that the basin of attraction of the zero solution includes Z and that solutions originating in Z will exhibit exponential decay. However, it is possible that the basin of attraction of the zero solution is actually much larger.

Another way in which we can determine the stability and asymptotic form of solution to equation (3.25) is to exploit this apparent decomposition of the solution given in equation (3.33). Define the functions u_1 and u_2 as solutions of

$$\begin{aligned} \partial_t u_1 &= \partial_x^2 u_1 + c\partial_x u_1 \\ \partial_t u_2 &= \partial_x^2 u_2 - c\partial_x u_2, \end{aligned} \tag{3.35}$$

with initial data satisfying

$$u_{1,2}(x,0) = u(x,0).$$

One can directly check that the solution to equation (3.25) is then given by

$$u(x,t) = \frac{u_1(x,t)}{1+e^{-cx}} + \frac{u_2(x,t)}{1+e^{cx}}. \tag{3.36}$$

The functions u_1 and u_2 represent the behavior of u in the far field near $+\infty$ and $-\infty$, respectively.

Because the two equations in (3.35) may be transformed into the heat equation, we may use the scaling variables of section 3.1 to determine their asymptotic expansions. To do so, define

$$\begin{aligned} u_1(x,t) &= \frac{1}{\sqrt{t+1}} w_1\left(\frac{x+c(t+1)}{\sqrt{t+1}}, \log(t+1)\right) \\ \eta_1 &= \frac{x+c(t+1)}{\sqrt{t+1}}, \quad \tau = \log(t+1), \\ u_2(x,t) &= \frac{1}{\sqrt{t+1}} w_2\left(\frac{x-c(t+1)}{\sqrt{t+1}}, \log(t+1)\right) \\ \eta_2 &= \frac{x-c(t+1)}{\sqrt{t+1}}, \quad \tau = \log(t+1). \end{aligned} \tag{3.37}$$

We then have

$$\partial_\tau w_{1,2} = \mathcal{L} w_{1,2}, \tag{3.38}$$

where $\mathcal{L} = \partial_\eta^2 + \frac{1}{2}\eta\partial_\eta + \frac{1}{2}$. We then obtain (in $L^2(m)$ for $m = 2$, as defined in equation (3.18))

$$w_{1,2}(\eta_{1,2},\tau) = \alpha_{1,2}(\tau)\phi_0(\eta_{1,2}) + \beta_{1,2}(\tau)\phi_1(\eta_{1,2}) + \mathcal{O}(e^{-\frac{3}{4}\tau}), \tag{3.39}$$

where

$$\alpha_{1,2}(\tau) = \int_{\mathbb{R}} w_{1,2}(\eta_{1,2},0)\,d\eta_{1,2}, \quad \beta_{1,2}(\tau) = e^{-\frac{1}{2}\tau}\int_{\mathbb{R}} \eta_{1,2} w_{1,2}(\eta_{1,2},0)\,d\eta_{1,2}. \tag{3.40}$$

Writing this result in terms of the original (x, t) variables, we find that

$$
\begin{aligned}
u_1(x,t) &= \left(\int u_1(x,0)dx\right)\frac{1}{\sqrt{4\pi(t+1)}}e^{-\frac{(x+c(t+1))^2}{4(t+1)}} \\
&\quad -\left(\int xu_1(x,0)dx\right)\frac{x+c(t+1)}{2(t+1)^{\frac{3}{2}}\sqrt{4\pi}}e^{-\frac{(x+c(t+1))^2}{4(t+1)}} + \mathcal{O}((t+1)^{-\frac{3}{4}}) \\
u_2(x,t) &= \left(\int u_2(x,0)dx\right)\frac{1}{\sqrt{4\pi(t+1)}}e^{-\frac{(x-c(t+1))^2}{4(t+1)}} \\
&\quad -\left(\int xu_2(x,0)dx\right)\frac{x-c(t+1)}{2(t+1)^{\frac{3}{2}}\sqrt{4\pi}}e^{-\frac{(x-c(t+1))^2}{4(t+1)}} + \mathcal{O}((t+1)^{-\frac{3}{4}})
\end{aligned}
\tag{3.41}
$$

Using equation (3.36), we may combine these two results to determine the asymptotic form of u. In doing so, we find

$$
u(x,t) = \frac{\int u(x,0)dx}{\sqrt{4\pi(t+1)}}\text{sech}(\frac{c}{2}x)e^{-\frac{c^2}{4}(t+1)}e^{-\frac{x^2}{4(t+1)}} + \mathcal{O}((t+1)^{-\frac{3}{4}}). \tag{3.42}
$$

This result is quite unexpected! The exponential factors in the definitions of u_1 and u_2 have combined with the leading order terms in the expansion to result in exponential (temporal) decay. The higher order corrections in the expansions for u_1 and u_2 can not be combined with the exponential factors, as their form is not known. Therefore, they become the new leading order terms, which decay only algebraically. Recall that these corrections come from the essential spectrum of the operator $\mathcal{L}$. Because the essential spectrum can be pushed further away from the imaginary axis by increasing the algebraic weight, *i.e.* working in $L^2(m)$ for larger m, the leading order algebraic decay rate of u can be increased by working in appropriately weighted spaces. For example, we have

$$
u(x,t) \sim \mathcal{O}((t+1)^{-\frac{(2m-1)}{4}}), \tag{3.43}
$$

if $u_0 \in L^2(m)$. A result of this type was previously known (see for example [35] and [60]), although a proof of this fact using scaling variables had not previously been given. Not only does this technique elucidate the geometric structure underlying the asymptotic decay of solutions, but it illustrates that the essential spectrum can sometimes play an important role in determining their behavior.

We have seen that, by considering equation (3.25) in an exponentially weighted space, we obtain exponential decay of solutions, while considering it in an algebraically weighted space we obtain algebraic decay of solutions. This result has to do with the advection term $\tanh(\frac{c}{2}x)u_x$, which transports data in toward zero. A nice intuitive explanation of this result is given in [60]. Consider an equation for which data flows in toward zero in a weighted space: $||u||_w = ||wu||_X$, where $w = w(x)$ is an weight function that increases as $|x|$ increases. Any mass that the solution has near infinity will initially experience a large weight, because

$w(x)$ is large when $|x|$ is large. As information gets transported in toward zero, the weight function decreases, thus causing the norm of the solution to decay in the weighted space. This generally leads to a decay rate given roughly by $\sup_x W(|x|)/W(|x|+ct)$. Hence, exponential weights lead to exponential decay, while algebraic weights lead to algebraic decay.

Another way to connect the two different weighted spaces is to notice that equation (3.34) is essentially the first term in equation (3.42). By studying the equation in this exponentially weighted space, we have effectively pushed the essential spectrum of $\mathcal{L}$ infinitely far away from the imaginary axis, thus removing all higher order corrections that decay only algebraically.

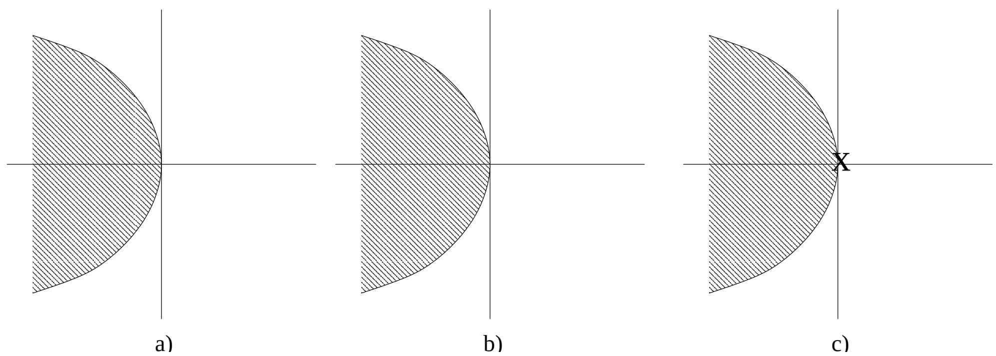

Figure 3·2: A schematic diagram of the spectrum of the operators A_2, A_3, and A_4 is shown in figures a), b), and c), respectively. In all three cases, the parabola is given by the set $\{\text{Re}\lambda = -\frac{(\text{Im}\lambda)^2}{c^2}\}$.

3.3 Example 3: $u_t = u_{xx} - c\tanh(\frac{c}{2}x)u_x$

The key property of the previous example was that data on the positive real line was transported to the left, and data on the negative real line was transported to the right. Thus, all information was transported in toward zero. This advective behavior allowed one to increase the algebraic, temporal decay rate of solutions by requiring that the initial data lie in an appropriately weighted L^2 space.

How will the behavior of solutions change if the direction of advection is reversed, *i.e.* if information is transported out toward infinity? This is the question we seek to answer using the following equation:

$$u_t = u_{xx} - c\tanh(\frac{c}{2}x)u_x. \tag{3.44}$$

By changing the sign of the advection term, the direction in which data is transported is reversed.

3.3.1 Spectral Properties

In order to determine the spectrum of the operator $A_3 = \partial_x^2 - c\tanh(\frac{c}{2}x)\partial_x$, we first note that the essential spectrum is given by the parabola $\{\lambda = -k^2 + ick : k \in \mathbb{R}\}$, or $\{\mathrm{Re}(\lambda) = -(\mathrm{Im}(\lambda))^2/c^2\}$. (See figure 3·2b.)

Next, we study the eigenvalue equation

$$(A_3 - \lambda\mathbf{1})u = u_{xx} - c\tanh(\frac{c}{2}x)u_x - \lambda u = 0. \tag{3.45}$$

By defining $v(x) = \int_{-\infty}^{x} \mathrm{sech}^2(\frac{c}{2}y)u(y)dy$, the above equation is transformed into equation (3.26), for which we have an explicit solution. Transforming back, we find that the solution to equation (3.45) is given by

$$\begin{aligned} u_\lambda(x) = k_0 \left(-\frac{c}{2}\mathrm{sinh}(\frac{c}{2}x) + \sqrt{(\frac{c^2}{4}+\lambda)}\mathrm{cosh}(\frac{c}{2}x)\right)\exp[\sqrt{(\frac{c^2}{4}+\lambda)}x] \\ + k_1 \left(-\frac{c}{2}\mathrm{sinh}(\frac{c}{2}x) - \sqrt{(\frac{c^2}{4}+\lambda)}\mathrm{cosh}(\frac{c}{2}x)\right)\exp[-\sqrt{(\frac{c^2}{4}+\lambda)}x]. \end{aligned} \tag{3.46}$$

This function will not decay at both $\pm\infty$ for any value of λ. In fact, it will grow exponentially at one of, or both, $x = \pm\infty$ for all λ except at $\lambda = 0$, where it will remain bounded. Therefore, we have that there are no eigenfunctions (in L^2).

Despite this fact, we claim that $\{\mathrm{Re}(\lambda) \leq -(\mathrm{Im}(\lambda))^2/c^2\} \subset \sigma(A_3)$. To see why this is the case, note that if A_3^* is the adjoint of A_3 and λ is an eigenvalue of A_3^*, then $\lambda \in \sigma(A_3)$ [15]. The reason for this is that, suppose $A_3^* u^* = \lambda u^*$, where u^* is a nonzero, bounded linear functional on L^2. For any $u \in L^2$ we then have

$$0 = (A_3^* - \lambda\mathbf{1})u^*(u) = u^*(A_3 u) - u^*(\lambda u) = u^*((A_3 - \lambda\mathbf{1})u).$$

Because this holds for any $u \in L^2$, this implies that u^* vanishes on the range of $A_3 - \lambda\mathbf{1}$. Because u^* is nonzero, this implies that there must be elements of L^2 that are not in the range of $A_3 - \lambda\mathbf{1}$. As a result, $A_3 - \lambda\mathbf{1}$ can not be invertible and, hence, $\lambda \in \sigma(A_3)$. Next, notice that $A_3^* = A_2$. Because every λ such that $\mathrm{Re}(\lambda) \leq -(\mathrm{Im}(\lambda))^2/c^2$ is an eigenvalue of A_2, these complex numbers are also elements of $\sigma(A_3)$. Hence, $\{\mathrm{Re}(\lambda) \leq -(\mathrm{Im}(\lambda))^2/c^2\} \subset \sigma(A_3)$.

To determine if this is all of the spectrum, consider

$$(A_3 - \lambda\mathbf{1})u = u_{xx} - c\tanh(\frac{c}{2}x)u_x - \lambda u = f. \tag{3.47}$$

The Green's function, $G_\lambda(x,y)$, for this equation can be found by solving

$$(A_{3,x} - \lambda\mathbf{1})G_\lambda = \delta(x-y)$$

using the method of variation of parameters. We find that

$$\begin{aligned} G_\lambda(x,y) = &-\frac{1}{2\lambda\sqrt{(\frac{c^2}{4}+\lambda)}}H(y-x)\left[-\frac{c}{2}\sinh(\frac{c}{2}x)+\sqrt{(\frac{c^2}{4}+\lambda)}\cosh(\frac{c}{2}x)\right]\cdot \\ &\left[\frac{c}{2}\text{sech}(\frac{c}{2}y)\tanh(\frac{c}{2}y)+\sqrt{(\frac{c^2}{4}+\lambda)}\text{sech}(\frac{c}{2}y)\right]\cdot e^{-\sqrt{(\frac{c^2}{4}+\lambda)}(y-x)} + \\ &-\frac{1}{2\lambda\sqrt{(\frac{c^2}{4}+\lambda)}}H(x-y)\left[+\frac{c}{2}\sinh(\frac{c}{2}x)+\sqrt{(\frac{c^2}{4}+\lambda)}\cosh(\frac{c}{2}x)\right]\cdot \\ &\left[-\frac{c}{2}\text{sech}(\frac{c}{2}y)\tanh(\frac{c}{2}y)+\sqrt{(\frac{c^2}{4}+\lambda)}\text{sech}(\frac{c}{2}y)\right]\cdot e^{-\sqrt{(\frac{c^2}{4}+\lambda)}(x-y)}. \end{aligned} \tag{3.48}$$

Analysis similar to that of the previous examples can be used to understand the behavior of this Green's function. Whenever $\text{Re}(\lambda) > -(\text{Im}(\lambda))^2/c^2$, the Green's function will decay exponentially as $|x|$ or $|y| \to \infty$. Hence, these values of λ are contained in the resolvent set. As a result, $\{\text{Re}(\lambda) \le -(\text{Im}(\lambda))^2/c^2\} = \sigma(A_3)$.

Remark 3.3.1 *In example 3, both $\lambda = 0$ and $\lambda = -\frac{c^2}{4}$ represent singularities of the Green's function: $\lambda = -\frac{c^2}{4}$ because it is a branch point of the square root function, and $\lambda = 0$ because λ appears in the denominator. Recall remark 3.2.2, which stated that for the previous example, $\lambda = -\frac{c^2}{4}$ was the only pole of the Green's function. For that example, we could obtain exponential decay at a rate given by $e^{-\frac{c^2}{4}t}$ by considering the equation in an exponentially weighted space. Working by analogy, by considering the current example in an exponentially weighted space, we would expect a decay rate of $e^{-0t} = 1$. In other words, we do not expect to be able to obtain an exponential decay rate for solutions to equation (3.44). We will see below that this is, in fact, the case.*

3.3.2 Behavior of Solutions

Because equation (3.44) can be transformed into equation (3.25) using the variable $v(x,t) = \int_{-\infty}^{x} \operatorname{sech}^2(\frac{c}{2}y)u(y,t)dy$, we can solve it explicitly. We find that

$$\begin{aligned} e^{A_3 t}u_0(x) = &-\frac{\frac{c}{2}}{\sqrt{4\pi t}}\sinh(\frac{c}{2}x)e^{-\frac{c^2}{4}t}\int_{\mathbb{R}} e^{-\frac{(x-y)^2}{4t}}\cosh(\frac{c}{2}y)v_0(y)dy \\ &-\frac{1}{\sqrt{4\pi t}}\cosh(\frac{c}{2}x)e^{-\frac{c^2}{4}t}\int_{\mathbb{R}}\frac{(x-y)}{2t}e^{-\frac{(x-y)^2}{4t}}\cosh(\frac{c}{2}y)v_0(y)dy, \end{aligned} \tag{3.49}$$

where $v_0(y) = \int_{-\infty}^{y} \operatorname{sech}^2(\frac{c}{2}s)u_0(s)ds$. Unfortunately, one cannot break the solution into pieces as in example 2. However, using the relationship between the two equations, the asymptotic form of solutions to equation (3.25), given in equation (3.42), can be used to determine the asymptotic form of solutions to equation (3.44). We compute

$$\begin{aligned} u(x,t) &= \cosh^2(\frac{c}{2}x)\partial_x\left[\frac{u_1(x,t)}{1+e^{-cx}} + \frac{u_2(x,t)}{1+e^{+cx}}\right] \\ &= \cosh^2(\frac{c}{2}x)\partial_x\left[\frac{2(\alpha_1+\alpha_2)}{\sqrt{4\pi(t+1)}}\operatorname{sech}(\frac{c}{2}x)e^{-\frac{x^2}{4(t+1)}}e^{-\frac{c^2}{4}(t+1)}\right] + \mathcal{O}((t+1)^{-1}) \\ &= \frac{2(\alpha_1+\alpha_2)}{\sqrt{4\pi(t+1)}}\cosh(\frac{c}{2}x)e^{-\frac{x^2}{4(t+1)}}e^{-\frac{c^2}{4}(t+1)}\left(-\frac{c}{2}\tanh(\frac{c}{2}x) - \frac{x}{2(t+1)}\right) + \mathcal{O}((t+1)^{-1}) \\ &= \frac{(\alpha_1+\alpha_2)}{\sqrt{4\pi(t+1)}}\left(-\frac{c}{2}\tanh(\frac{c}{2}x) - \frac{x}{2(t+1)}\right)\left(e^{-\frac{(x+c(t+1))^2}{4(t+1)}} + e^{-\frac{(x-c(t+1))^2}{4(t+1)}}\right) \\ &\quad + \mathcal{O}((t+1)^{-1}). \end{aligned}$$

Notice that the exponentially decaying (in time) terms have been absorbed by the factor $\cosh^2(\frac{c}{2}x)$. As a result, the leading order term in the expansion for u will decay like $\frac{1}{\sqrt{t+1}}$ regardless of in which algebraically weighed space, $L^2(m)$, the initial data lies.

We see that, while in example 2 the temporal decay rate could be increased by changing the algebraic weight of the L^2 space, that is not the case for example 3. The reason for this is the advection term. As explained in the previous section, for so-called "inflowing" equations, in which data is transported in toward the origin, the temporal decay rate of solutions can be increased in this manner. On the other hand, for "outflowing" equations such as example 3, this is not the case. The intuitive reasoning is as follows [60].

Consider an equation for which data flows out away from zero in a weighted space: $||u||_w = ||wu||_X$, where $w = w(x)$ is a weight function that increases as $|x|$ increases. Suppose the initial data has a some amount of mass near zero, but an arbitrarily small amount near infinity. As the data gets transported away from zero, the initial mass will experience an increasing weight, thus causing the norm of the solution to grow, or at best remain constant (depending on the other dynamics of the PDE), in the weighted space.

One may think that instead of an increasing weight, a weight that decays to zero at infinity could be used to increase the temporal decay rate of solutions. Such a technique could work for linear equations, but will create problems for nonlinear analysis, where any weight function must be bounded from below in order to perform certain estimates.

Note that this argument applies to exponential weights, as well. This confirms the suggestion of remark 3.3.1, that one would be unable to to obtain an exponential decay rate of solutions by using an exponential weight.

3.4 Example 4: $u_t = u_{xx} + c\tanh(\frac{c}{2}x)u_x + \frac{c^2}{2}\text{sech}(\frac{c}{2}x)u$, linear stability of traveling fronts in Burgers equation

Finally, we consider the linear equation that one obtains from linearizing around the traveling wave in Burgers equation:

$$u_t = u_{xx} + c\tanh(\frac{c}{2}x)u_x + \frac{c^2}{2}\text{sech}(\frac{c}{2}x)u. \tag{3.50}$$

Note that this equation is simply the derivative of equation (3.25). If we define $v(x,t) = \int_{-\infty}^{x} u(y,t)dy$, then v satisfies $v_t = v_{xx} + c\tanh(\frac{c}{2}x)v_x$, which is equation (3.25). As a result, analysis of this equation follows almost immediately from the results of section 3.2. The reason why it is included as a separate example is because this equation is exactly the linear operator one obtains after linearizing around the traveling front (with wavespeed c) in Burgers equation.

3.4.1 Spectral Properties

Using the results of section 3.2, one can see that the spectrum is identical to that of example 2, with an additional eigenvalue at $\lambda = 0$ (see figure 3·2c). The eigenfunction associated to the zero eigenvalue is given by $\text{sech}^2(\frac{c}{2}x)$. The reason why zero is not an eigenvalue of the operator A_2 is that $\int_{-\infty}^{x}\text{sech}^2(\frac{c}{2}y)dy = \tanh(\frac{c}{2}x) \notin L^2$. The relevance of this difference in spectrum will be further discussed in chapter 6.3 below.

3.4.2 Behavior of Solutions

Again using the results from section 3.2, we see that the asymptotic form of solutions to equation (3.50) is

$$u(x,t) \sim \partial_x \left[\frac{\int \left(\int_{-\infty}^{x} u(y,0)dy \right) dx}{\sqrt{4\pi(t+1)}} e^{-\frac{x^2}{4(t+1)}} e^{-\frac{c^2}{4}(t+1)} + \mathcal{O}((t+1)^{-\frac{m}{2}+\frac{1}{4}}) \right], \tag{3.51}$$

for initial data satisfying $\int_{-\infty}^{x} u(y,0)dy \in L^2(m)$. Thus, as in example 2, the temporal decay rate of solutions may be increased by requiring that initial data lie in an appropriate function space. This results from the fact that equation (3.50), like equation (3.25), is an "inflowing" equation, as a result of the advection term.

Chapter 4

Example: locating the spectrum in a model of bioremediation

As mentioned in chapter 2, one difficulty that can arise when studying the stability of stationary solutions to PDEs is that it can be quite difficult simply to calculate the spectrum of the linearization. In this chapter we focus on such an example, in which the stability of a traveling wave solution in a model of bioremediation, a process for cleaning contaminated soil, is analyzed.

The main tool that we will use in the stability analysis of the traveling wave is known as the Evans function. It is an analytic function $D(\lambda) : \Omega \subset \mathbb{C} \to \mathbb{C}$ that is zero exactly if λ is an eigenvalue of the linearized operator. Although one can always write down an expression for the Evans function, it is, in general, difficult to determine the zeros of the function.

As we will see below, the traveling wave solution of the bioremediation model has a fast-slow structure consisting of multiple time scales. The associated Evans functions will inherit this structure, and, as a result, we will be able to determine its zeros and the stability of the wave.

We will begin this chapter with a brief introduction to the Evans function, followed by an introduction to the bioremediation model. In the third section, geometric singular perturbation theory will be used to construct the traveling wave solution [3]. The fourth section contains the construction of the Evans function for the associated stability problem. In addition, the properties of the wave, found in the previous section, will be used to locate its zeros and, hence, determine the linear stability of the wave.

4.1 The Evans Function

Consider a general, nonlinear PDE

$$U_t = AU + N(U),$$

where $U = U(x,t) : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}^n$ is a vector valued function. Suppose that we are interested in studying the stability of a stationary solution $f(x)$ that exponentially approaches some constants as $x \to \pm\infty$. If we define $U(x,t) = f(x) + u(x,t)$, then the perturbation u

satisfies

$$u_t = Au + DN(f(x))u + F(u,x),$$

where the function F is some nonlinear term. The linear operator for the stability problem is then given by $\mathcal{L} = A + DN(f(x))$. Although this operator is dependent upon the spatial variable x, using theorem 2.1.3 one can typically determine its essential spectrum. Let's suppose that this is the case and that $\sigma_{ess}(\mathcal{L}) \subset \{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) \leq -\delta < 0\}$. In order to compute the point spectrum of the wave, we must determine if there are any eigenvalues of $\mathcal{L}$ in $\Omega = \{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) > -\delta\}$. Thus, we must study the eigenvalue equation

$$\lambda u = Au + DN(f(x))u,$$

which is a nonautonomous ODE.

In order to fix ideas, let's assume that the operator A is a constant coefficient operator of the form $A = D\partial_x^2 + M\partial_x$, where D and M are $n \times n$ constant matrices and D is invertible. The eigenvalue ODE may then be written as a first order system of ODEs

$$\partial_x \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ D^{-1}\left(\lambda - DN(f(x))\right) & -D^{-1}M \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} \equiv B(x;\lambda) \begin{pmatrix} u \\ v \end{pmatrix}, \tag{4.1}$$

where $(u,v) \in \mathbb{C}^{2n}$. A element $\lambda \in \Omega$ will be an eigenvalue if the corresponding solution u of the above equation lies in the Banach space X. Although we have not specified what X is, we note that for most choices of X, *e.g.* L^p or C^0, the function u must at least be bounded as $|x| \to \infty$. The behavior of (u,v) as $|x| \to \infty$ will be governed by the asymptotic matrices

$$B^{\pm}(\lambda) = \lim_{x \to \pm\infty} B(x;\lambda). \tag{4.2}$$

One can directly check that for $\lambda \notin \sigma_{ess}(\mathcal{L})$, the matrices $B^{\pm}$ will both be hyperbolic, meaning that none of their eigenvalues will have zero real part. As a result, for any value of $\lambda \in \Omega$, the solution (u,v) to equation (4.1) will either decay exponentially to zero as $|x| \to \pm\infty$, or grow exponentially at one, or both, ends. Thus, any value of $\lambda \in \Omega$ will be an eigenvalue of $\mathcal{L}$ if and only if the corresponding solution to equation (4.1) decays exponentially to zero as $|x| \to \pm\infty$.

Remark 4.1.1 *The terminology can sometimes be a bit confusing, because we must refer to eigenvalues of both the original linear operator $\mathcal{L}$ and the matrices $B^{\pm}$. In order to clarify things, the eigenvalues of $\mathcal{L}$ are often referred to as temporal eigenvalues (because they control the behavior of u for large time) and the eigenvalues of $B^{\pm}$ as spatial eigenvalues (because they control the behavior of u for large values of the spatial variable $|x|$). If the* *context is clear, we will refer to both simply as eigenvalues.*

The Evans function $D(\lambda)$ was developed as a tool for determining exactly when such bounded solutions exist. It is defined in such a way so that $D(\lambda) = 0$ if and only if there exists a solution to equation (4.1) that decays to zero asymptotically in space. It was first introduced in a series of papers by John Evans [16]-[19] in the 1970s. A concise description was later given in [53] and [54], and a rigorous development of the Evans function in the context of geometric singular perturbation theory can be found in [1]. A nice review of the use of the Evans function to determine the stability of traveling waves is given in [51].

The Evans function is defined as follows. Suppose that, for some $\lambda \in \Omega$, $B^{\pm}$ have k eigenvalues with negative real part and $2n - k$ eigenvalues with positive real part. Note that, because $B^{\pm}$ are hyperbolic for $\lambda \in \Omega$, this must be true for all $\lambda \in \Omega$. The unstable eigenvalues of B^- will be denoted by ν_j^- for $j = 1 \cdots 2n{-}k$, and the stable eigenvalues of B^+ will be denoted by ν_l^+ for $l = 1 \cdots k$. Let $\{e_1^+(\lambda), \ldots, e_k^+(\lambda)\}$ be a basis for the k-dimensional stable subspace of B^+, and let $\{e_1^-(\lambda), \ldots, e_{2n-k}^-(\lambda)\}$ be a basis for the $2n-k$-dimensional unstable subspace of B^-. Standard results regarding solutions to nonautonomous ODEs [12] show that there exists a k-dimensional subspace of solutions $\Phi_+(x;\lambda)$ and a $2n-k$-dimensional subspace of solutions $\Phi_-(x;\lambda)$ to equation (4.1) such that

$$\begin{aligned}
\Phi_-(x;\lambda) &= \text{span}\{\phi_1^-(x;\lambda), \ldots, \phi_{2n-k}^-(x;\lambda)\}, \quad & \phi_j^-(\lambda) \sim e_j^-(\lambda)e^{\nu_j^-(\lambda)x} \text{ as } x \to -\infty \\
\Phi_+(x;\lambda) &= \text{span}\{\phi_1^+(x;\lambda), \ldots, \phi_k^+(x;\lambda)\}, \quad & \phi_l^+(\lambda) \sim e_l^+(\lambda)e^{\nu_n^+(\lambda)x} \text{ as } x \to +\infty.
\end{aligned}$$

A solution to equation (4.1) that decays to zero at both $+\infty$ and $-\infty$ will lie in the intersection of these two subspaces. This intersection will be nonempty if the determinant of the $2n \times 2n$ matrix $[\Phi_-(x;\lambda), \Phi_+(x;\lambda)]$ is zero. The Evans function is then defined to be

$$D(\lambda) = e^{-\int_0^x \text{Tr}B(s;\lambda)ds}\det[\Phi_-(x;\lambda), \Phi_+(x;\lambda)]. \tag{4.3}$$

One can see from this definition that the Evans function will be zero when an intersection of the subspaces Φ_+ and Φ_- exists. Notice that the notation implies that the Evans function is independent of the spatial variable x. To see that this is the case, one can compute

$$\begin{aligned}
\frac{d}{dx}D(\lambda) &= -\text{Tr}BD(\lambda) + e^{-\int_0^x \text{Tr}B(s;\lambda)ds}\left(\frac{d}{dx}\det[\Phi_-(x;\lambda), \Phi_+(x;\lambda)]\right) \\
&= -\text{Tr}BD(\lambda) + \text{Tr}BD(\lambda) \\
&= 0.
\end{aligned}$$

In addition, this calculation demonstrates why there is an exponential factor in the definition of $D(\lambda)$ - otherwise the Evans function would be dependent upon the spatial variable.

Remark 4.1.2 *There is an alternate way to define the Evans function, in which the sub-*

space Φ_- (or Φ_+) is defined in terms of solutions to the adjoint equation associated to equation (4.1). Using this formulation, the exponential factor in equation (4.3) is not necessary, because the determinant will automatically be independent of x.

As mentioned above, although a formula exists for the Evans function, it is often difficult to determine the zeros of the function. In some examples, however, the structure of the PDE provides a mechanism for doing so. One class of equations for which this is the case is singularly perturbed PDEs, and a specific example is given by the bioremediation model below. Other examples of singularly perturbed equations in which the Evans function is used to locate the spectrum of a linear operator include, but are not limited to, [13], [14], [27], and [33].

4.2 The Bioremediation Model

We now introduced the PDE model of bioremediation, which will serve as an example illustrating the difficulties that can arise in locating the spectrum of a linear operator.

In situ bioremediation is a promising technique for cleaning contaminated soil (see [43] and the references therein). The process typically involves an organic pollutant (labeled as a substrate), a nutrient (labeled as an electron acceptor), and indigenous microorganisms. Roughly speaking, when both the substrate and acceptor are present, the microorganisms consume the acceptor and degrade the substrate, decontaminating the soil. Bioremediation involves complex interactions and has many controlling factors which make it difficult to understand. Mathematical analysis of simplified models may allow for the identification of key components which control the behavior of the system, allowing for more effective implementation.

We study the nondimensional form of the Oya-Valocchi bioremediation model in [43], [40], and [59]. This is a conceptual model that was developed in order to better identify key controlling factors, and also to help connect laboratory work with field experiments [43]. The situation is idealized to be a one-dimensional, semi-infinite soil column with initial constant background level of substrate and of biomass of the microorganisms, but no acceptor. Beginning at time $t = 0$, a constant level of acceptor is injected continuously at the surface of the soil column. This creates a concentration profile of the acceptor that is a traveling front propagating down the soil column, connecting the positive (injection) concentration behind the front and the zero concentration ahead of it. By contrast, the traveling profile of the substrate concentration connects the zero (completely remediated) level behind the front to the constant (initial, undisturbed) level ahead of it. Moreover, the substrate front lags slightly behind that of the acceptor, so that there is a region of overlap between the fronts. In this region, known as the biologically active zone (BAZ), the microbial population is highly elevated due to the supply of both nutrient and substrate. As the fronts move downstream, the location of elevated microbial population moves with them, and after the fronts pass a given location the biomass population returns to its equilibrium level. Thus, the biomass concentration exhibits a single bump profile which travels with the fronts as the reaction progresses down the column. See figure 4·1.

Two structural assumptions are incorporated into the model. First, the microbes are attached to particles in the soil and therefore do not move. Second, the acceptor is non-sorbing, meaning it travels through the column at the pore water velocity (which has been normalized to be 1), whereas the substrate is sorbing, traveling at the retarded velocity $\frac{1}{R_d}$ where the retardation factor satisfies $R_d > 1$.

Mathematically, we let $S = S(x,t)$, $A = A(x,t)$, and $M = M(x,t)$ denote the concentrations of the substrate, acceptor, and microorganisms, respectively. The model equations we study are

$$\begin{aligned} R_d \frac{\partial S}{\partial t} - \frac{\partial^2 S}{\partial x^2} + \frac{\partial S}{\partial x} &= -a_1 f_{bd}(S, A, M) \\ \frac{\partial A}{\partial t} - \frac{\partial^2 A}{\partial x^2} + \frac{\partial A}{\partial x} &= -a_1 a_2 f_{bd}(S, A, M) \\ \frac{\partial M}{\partial t} &= a_3 f_{bd}(S, A, M) - a_4(M - 1) \\ f_{bd} &= M \left(\frac{S}{K_S + S} \right) \left(\frac{A}{K_A + A} \right), \end{aligned} \tag{4.4}$$

for $x \in \mathbb{R}$, $t > 0$, in which the diffusion coefficients have been scaled to 1 (see remark 4.3.3). Because we are interested in traveling waves, the asymptotic conditions are

$$\begin{aligned} &S(-\infty, t) = 0 \quad A(-\infty, t) = 1 \quad M(-\infty, t) = 1 \\ &S(+\infty, t) = 1 \quad A(+\infty, t) = 0 \quad M(+\infty, t) = 1. \end{aligned} \tag{4.5}$$

The reaction function f_{bd} represents Monod reaction kinetics. The magnitude of this reaction function is directly proportional to the product of the substrate and acceptor concentrations. Moreover, there is a saturation effect controlled by the parameters K_S and K_A. These are referred to as the relative half-saturation constants for the substrate and acceptor and indicate the degree to which the presence of each (or lack thereof) may limit the growth of the microorganisms. For example, if $K_S \ll 1$, then $\frac{S}{K_S+S} \approx 1$ and the substrate has little effect on microbial growth in the reaction zone, except near the trailing edge of the substrate front where $S < K_S$. However, if $K_S \gg 1$, then $\frac{S}{K_S+S}$ is small and everywhere the substrate limits microbial growth. Thus, the magnitudes of these quantities will be important in determining the dynamics of the reaction.

The parameters a_i represent ratios of various timescales of the reaction. As explained in [43], a_1 represents the ratio between the transport timescale and the biodegradation timescale of substrate. Similarly, the combined parameter $a_1 a_2$ represents the corresponding ratio for the acceptor. The parameters a_3 and a_4 are the ratios of the transport timescale to the maximum cell growth and cell decay of the microorganisms, respectively.

Finally, the asymptotic conditions (4.5) may be explained as follows. The asymptotic conditions at $-\infty$ represent the fact that, behind the BAZ, the substrate has been completely degraded, the acceptor level is equal to its injection level, and the microorganism population has returned to its equilibrium level. At $+\infty$, ahead of the BAZ, the soil remains undisturbed and contaminated, and thus the substrate, acceptor, and microorganisms are all equal to their initial levels.

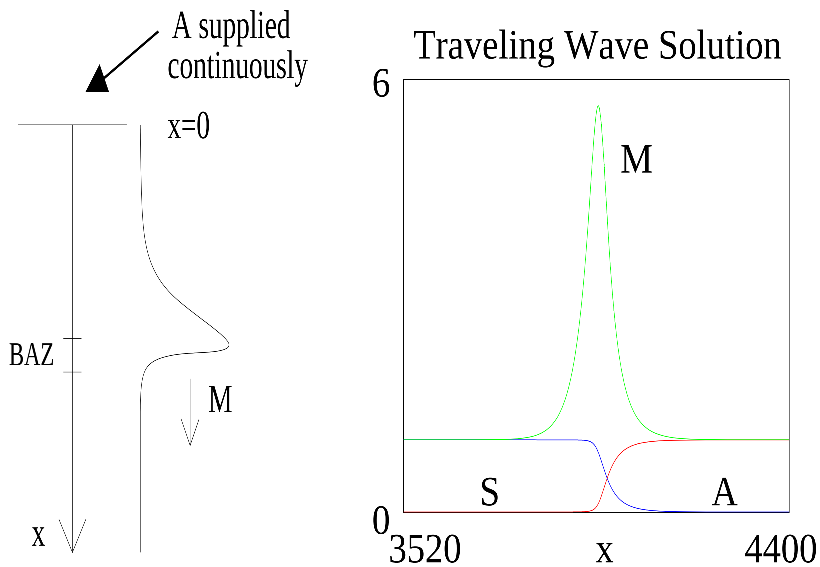

Figure 4·1: A schematic diagram of the soil column and the M component of the traveling wave is shown on the left. The traveling wave, observed in numerical simulations in [43] with $a_1 = 0.011$, $a_2 = 0.345$, $a_3 = 0.0885$, $a_4 = 0.0218$, $R_d = 3$, and $K_S = K_A = 0.3$, is shown on the right. S increases from 0 to 1, while A decreases from 1 to 0.

It is of interest to note that the boundary and initial conditions corresponding to the soil column experiment (for $x \in [0, \infty)$ and $t > 0$) are

$$\left(-\frac{\partial S}{\partial x} + S\right)_{x=0} = 0 \quad \left(-\frac{\partial A}{\partial x} + A\right)_{x=0} = 1$$
$$S(x,0) = 1 \quad A(x,0) = 0 \quad M(x,0) = 1, \tag{4.6}$$

and that one should use these to study the initial formulation of the traveling wave. In this case, the boundary conditions for S and A at $x = 0$ represent the fact that a constant level of acceptor is injected continuously there, while no substrate is added to the system. Here, as written above, we assume the traveling wave has already formed.

Traveling waves of the type shown in figure 4·1 have been investigated in [36], [40], [43],and [59]. A dimensional version of (4.4) was developed in [41], [42], and [57], and further studied in [43], wherein the nondimensional version (4.4) was derived. These authors carried out an extensive numerical investigation of the model, discovering the traveling wave. In addition, they investigated the effects of varying the parameters K_S and K_A, noting that for some values the traveling wave is stable, while for others there is a stable, time-periodic traveling wave. In this work, the authors were the first to exploit mathematically traveling waves in a bioremediation model in order to determine the substrate removal rate.

Further analysis of the dimensional model was carried out in [40]. In this work, the authors proved the existence the traveling wave for $R_d > 1$, and positive values of the other parameters. Working with an elliptically regularized form of the dimensional model on a finite domain and using topological degree theory, the authors construct the traveling wave as the fixed point of an appropriate map. Existence is then proven by extending the result to the original, non-elliptic model on the infinite line. In the process, bounds on all three components of the solution are obtained, in particular, an explicit bound on the peak height of the M pulse in terms of the dimensional parameters.

In [59], the transition between the traveling wave behavior and the time-periodic traveling wave behavior, first discovered in [43], is investigated. The authors study the dimensional model in the absence of diffusion using a relaxation procedure and WKB analysis. Using a reduced, two-component model, the authors explicitly determine the traveling wave and show it is stable for certain parameter values, losing stability in an oscillatory fashion.

Ultimately we would like to gain a better mathematical understanding of the mechanism which causes this loss of stability. A geometric construction of the traveling wave will help in this understanding. The geometric structures underlying a traveling wave solution, and how these structures vary with the parameters, provide direct insight into mechanisms governing its stability.

In section 4.3, we provide a geometric construction of the traveling wave solution for sufficiently large (relative to a singular perturbation parameter δ) values of the half saturation constants, K_S and K_A. In this parameter regime it will be shown that the entire traveling wave lies on a three-dimensional slow manifold within the five-dimensional phase space of the traveling wave ODE system. Within this slow manifold, the wave will be constructed in the transverse intersection of appropriate stable and unstable manifolds.

In addition to providing further insight into the bioremediation model, this construction is mathematically interesting because of the nonlinear reaction term f_{bd}. Two components of the function, $(\frac{S}{S+K_S})$ and $(\frac{A}{A+K_A})$, have derivatives which become large as K_S and K_A become small. In other words, because the half saturation constants are scaled so that $K_{S,A} \to 0$ as $\delta \to 0$, the reaction term is not uniformly bounded in the C^1 topology as $\delta \to 0$. This prevents a direct application of Fenichel theory [20], thus preventing one from concluding that geometric structures which are present in the phase space of the model in the asymptotic limit persist. In order to overcome this difficultly, we change coordinates by compactifying the S and A directions in a manner that naturally reflects the components of the reaction function.

As mentioned above, this construction is only valid for large values of the half saturation constants (relative to δ). For small values of the half saturation constants it will be shown that a different analysis is necessary. As we will see, this is because the geometry of the phase space changes significantly as the half saturation constants decrease through a critical scaling with respect to δ. The parameter regime including smaller values of the half saturation constants is where the bifurcation to a periodic traveling wave has been observed numerically. Using the moving grid scheme in [4], in section 4.3.4 we will explore numerically this parameter regime, including the bifurcation, and discuss how the geometry of the phase space changes in this case.

Section 4.4 will consist of the stability analysis and will use results from the existence construction of section 4.3. For sufficiently large values of the half-saturation constants we will indicate how to show that the wave is (spectrally) stable.

4.3 Construction of Traveling Wave Solution

In this section, we present a construction of the traveling wave using geometric singular perturbation theory. In order to better understand why the properties of the solution itself play a key role in determining the properties of the associated linear operator, consider a general, parabolic PDE:

$$u_t = \mathcal{F}(u),$$

and suppose we are interested in studying the stability of a stationary solution $u(x,t) = \varphi(x)$, which is a solution to the equation

$$0 = \mathcal{F}(\phi). \tag{4.7}$$

We write $u(x,t) = \varphi(x) + v(x,t)$, where v is assumed to be small, and determine the equation of evolution for v. We find

$$v_t = D\mathcal{F}(\varphi(x))v + \mathcal{N}(v,x),$$

where $D\mathcal{F}(\varphi(x))$ is the linearization of $\mathcal{F}$ about the solution of interest, $\varphi(x)$. We see from this equation that the structure of the linear operator governing the evolution of v is directly dependent upon the structure of the underlying solution. In particular, consider the eigenvalue equation associated to the linear stability problem:

$$\lambda v = D\mathcal{F}(\varphi(x))v. \tag{4.8}$$

Solutions to this equation live in the tangent space of solutions to equation (4.7) and, hence, the behavior of the stationary solution itself will play a key role in determining the stability properties of the wave. As a consequence, before analyzing the stability of the traveling wave solution to the bioremediation model, we present a geometric construction of the wave. The properties of the solution found in this construction will then be used to analyze the associated Evans function, thus determining the stability of the wave.

The remainder of this section is organized as follows. In subsection 4.3.1, we reformulate the model in terms of a moving coordinate frame which is appropriate to the study of traveling waves. In addition, we derive scalings of the parameters based on the numerical values used in [43] in terms of a small quantity δ, placing the problem within the context of geometric singular perturbation theory. In subsection 4.3.2, we present the geometric construction of the traveling wave solution for sufficiently large values of the half saturation constants. In subsection 4.3.3, we show that the asymptotics for the traveling wave solution agree with the results obtained from numerical simulations, where we note that the asymptotics have been carried out to include both the leading order terms and the first-order corrections. Finally, in subsection 4.3.4, we investigate numerically the bifurcation the traveling wave undergoes for small values of the half saturation constants.

4.3.1 Scalings for traveling waves

We are interested in traveling wave solutions representing an advancing front for the acceptor A, a trailing front for the substrate S, and a pulse for the biomass M. Plugging the Ansatz

$$s = S(x - ct), \quad a = A(x - ct), \quad m = M(x - ct)$$

into (4.4), we find that the system becomes

$$\begin{aligned} s'' + (cR_d - 1)s' &= a_1 f_{bd} \\ a'' + (c - 1)a' &= a_1 a_2 f_{bd} \\ cm' &= a_4(m - 1) - a_3 f_{bd}. \end{aligned} \tag{4.9}$$

where $'$ denotes differentiation with respect to the moving coordinate $\xi \equiv x - ct$. Note that c represents the speed of the wave. With respect to this new coordinate, the wave is now a stationary solution. The asymptotic conditions are

$$\begin{aligned} s(-\infty) = 0, &\quad s(+\infty) = 1 \\ a(-\infty) = 1, &\quad a(+\infty) = 0 \\ m(-\infty) = 1, &\quad m(+\infty) = 1. \end{aligned} \tag{4.10}$$

Note that these conditions also imply that $s'(\pm\infty) = a'(\pm\infty) = m'(\pm\infty) = 0$.

From the s and a equations in (4.9), one may compute analytically the wave speed c. To do this, eliminate the term f_{bd} from the equations and integrate once with respect to ξ. Using the asymptotic conditions, we find

$$c = \frac{a_2 + 1}{a_2 R_d + 1}. \tag{4.11}$$

As mentioned in [43] and [40], the wave speed depends only on the relative rates of consumption of the substrate and acceptor by the microorganisms (a_2), and the amount of sorbtion of the substrate (R_d). The wave speed is independent of both the microbial parameters (a_3 and a_4), and also of a_1.

Traveling waves are observed numerically in [43] for the following parameter values:

$$a_1 = 0.011 \quad a_2 = 0.3450 \quad a_3 = 0.0885 \quad a_4 = 0.0218 \quad R_d = 3.0, \tag{4.12}$$

and a range of values of K_S and K_A. The differences in magnitudes of these parameters suggest that we introduce a small parameter δ and scale the parameters in terms of this quantity. In turn, this will allow us to use singular perturbation theory to determine the mechanisms which produce the observed traveling wave behavior. Moreover, the above values suggest rescaling the parameters as

$$a_1 = \delta^2 \tilde{a}_1 \quad a_2 = a_2 \quad a_3 = \delta \tilde{a}_3 \quad a_4 = \delta^2 \tilde{a}_4 \quad R_d = R_d. \tag{4.13}$$

Here $\tilde{a}_1$, $\tilde{a}_3$ and $\tilde{a}_4$ are assumed to be $\mathcal{O}(1)$ with respect to δ.

From numerical simulations, we see that the properties of the traveling wave depend significantly on the half saturation constants K_S and K_A, and we are interested in a range of values. Specifically, we scale them as

$$K_S = \delta^\kappa \tilde{K}_S \quad K_A = \delta^\kappa \tilde{K}_A. \tag{4.14}$$

Inserting the rescaled quantities into (4.9) and writing the equations as a system of five, first-order equations, we obtain

$$\begin{aligned} s' &= v \\ v' &= -(cR_d - 1)v + \delta^2 \tilde{a}_1 f_{bd} \\ a' &= r \\ r' &= -(c-1)r + \delta^2 \tilde{a}_1 a_2 f_{bd} \\ m' &= \delta^2 \frac{\tilde{a}_4}{c}(m-1) - \delta \frac{\tilde{a}_3}{c} f_{bd}. \end{aligned} \tag{4.15}$$

These are the equations we will analyze throughout the rest of the paper.

We will construct the traveling wave for $0 < \kappa < 1$ in section 4.3.2, examine its properties in section 4.3.3, and report on numerical simulations in the regime $\kappa \geq 1$, in which a Hopf bifurcation takes place, in section 4.3.4. The approach in section 4.3.2 can be extended naturally to include the threshold cases $\kappa = 0$ and $\kappa = 1$, although certain technical details are different. In the simulations of [43] the case $\kappa < 0$, ie K_S, $K_A \gg 1$, has been considered briefly. The waves observed in this case can also be constructed using an analytical approximation procedure, which is in fact more straightforward since the reaction term is regular (in the C^1 topology) as $\delta \to 0$. We do not consider this case in any further detail here.

Remark 4.3.1 *We have also explored other scalings, such as setting* $a_2 = \sqrt{\delta}\tilde{a}_2$ *and* $R_d =$

$\frac{1}{\sqrt{\delta}}\tilde{R}_d$, *as would be suggested by taking* $\delta = 0.1$*; however, there are various reasons for assuming both are* $\mathcal{O}(1)$*. These include keeping the reaction terms in the equations for* v *and* r *in (4.15) of the same order and also keeping the terms* $(cR_d - 1)$ *and* $(c - 1)$ *in (4.15) of the same order. In addition, this allows us to retain as many terms as possible, ie to find a significant degeneration of the system.*

Remark 4.3.2 *In both [43] and [40], it was shown that the assumption that the retardation factor satisfies* $R_d > 1$ *is crucial for the existence of traveling waves. Biologically this can be understood as providing a mechanism to increase the width of the BAZ. Because the substrate is sorbing, it lags behind after the initial injection of the acceptor before the substrate front begins moving downstream with that of the acceptor. If the retardation factor was not present (ie if the substrate was not sorbing) then this lag would not occur and the BAZ would be too narrow to allow an appreciable amount of substrate to be degraded as the reaction progressed downstream.*

From the above analysis, one can begin to see mathematically why it is necessary that at least $R_d \neq 1$*. If* $R_d = 1$*, then we would have* $c = 1$*, and hence the quantities* $(c - 1)$ *and* $(cR_d - 1)$ *in equation (4.15) would both be* 0*. Therefore, if the substrate was not sorbing, the advection terms would effectively drop out, which would dramatically change the following analysis and also the observed dynamics.*

Remark 4.3.3 *We have implicitly chosen the diffusion coefficients in system (4.4) to be equal and scaled them to* 1*. It may be possible to extend our analysis to the case where the are not equal. In addition, in both [43] and [40], the dimensional model with zero diffusion was investigated. It was shown that, in this case, a traveling wave solution still exists. It is interesting to note that setting the dimensional diffusion coefficient to zero is equivalent to setting the parameters* a_1*,* a_3*, and* a_4 *equal to zero, while also rescaling space* (x) *and time* (t)*. See the transformation between the dimensional and nondimensional coordinates in [43] for more details.*

4.3.2 Geometric construction of the traveling wave

The goal of this section is to prove the following theorem:

Theorem 4.3.4 *There exists a* $\delta_0 > 0$ *such that for all* $\delta \in (0, \delta_0)$*, for all* $\kappa \in (0, 1)$*, and for all* $\tilde{a}_1$*,* a_2*,* $\tilde{a}_3$*,* $\tilde{a}_4$*,* $\tilde{K}_S$*,* $\tilde{K}_A$*, and* R_d $\mathcal{O}(1)$ *and positive, system (4.15) has a traveling wave solution,* $\gamma_{tw}(\xi) = (s_{tw}, v_{tw}, a_{tw}, r_{tw}, m_{tw})(\xi)$*, connecting* $(s, v, a, r, m) = (0, 0, 1, 0, 1)$ *at* $-\infty$ *with* $(s, v, a, r, m) = (1, 0, 0, 0, 1)$ *at* $+\infty$*. In addition, let*

$$\mathcal{L}^- = \{(s, a, m) = (0, 1, m) \quad | \quad m \in [1, 1 + \delta^{-1}\frac{\tilde{a}_3(R_d - 1)}{\tilde{a}_1(a_2 + 1)}]\},$$

and

$$\mathcal{I} = \{(s, a, m) = (s, 1 - s, 1 + \delta^{-1} \frac{\tilde{a}_3(R_d - 1)}{\tilde{a}_1(a_2 + 1)}(1 - s) \quad | \quad s \in [0, 1]\}.$$

Then the s, a, and m components of the traveling wave are $\mathcal{O}(\delta)$ close to $\mathcal{S}_\delta \equiv \mathcal{L}^- \bigcup \mathcal{I}$.

The theorem states that, to leading order, the traveling wave is the union of $\mathcal{L}^-$ and $\mathcal{I}$. These two curves can be understood biologically as follows. $\mathcal{L}^-$ corresponds to the portion of the traveling wave to the left of the BAZ, where s and a are equal to their asymptotic values at $-\infty$ and m is slowly decaying to its equilibrium value (as $\xi \to -\infty$). Moreover, the constant $1 + \delta^{-1} \frac{\tilde{a}_3(R_d - 1)}{\tilde{a}_1(a_2+1)}$ is, to leading order, the maximum value of the microorganism population.

The curve $\mathcal{I}$ corresponds to the portion of the traveling wave inside the BAZ, in which m decays from its maximum value to its asymptotic value at $+\infty$, and s and a transition from their asymptotic values at $-\infty$ to those at $+\infty$. $\mathcal{I}$ also contains the portion of the wave that lies to the right of the BAZ, namely, the point $(1, 0, 1)$.

As we will see below, these two pieces of the traveling wave correspond to portions of the wave that evolve on different time scales. The dynamics on $\mathcal{I}$ occurs on a slow, $\mathcal{O}(\delta)$, timescale, and the dynamics on $\mathcal{L}^-$ occurs on a super-slow, $\mathcal{O}(\delta^2)$, timescale. This separation of timescales is due to the fact that the timescale of the reaction, which is governed by the magnitude of the reaction function $\delta \tilde{a}_i f_{bd}$, is different than the timescale of the intrinsic dynamics of the microorganisms, given by the parameter $\delta^2 \tilde{a}_4$.

It will be shown below that there exists both a slow and a super-slow invariant manifold in the phase space of (4.15). The leading order slow system is integrable, and $\mathcal{I}$ corresponds to one of the integral curves. The leading order super-slow dynamics consist of invariant lines in the phase space where the only dynamic variable is m. The curve $\mathcal{L}^-$ is one of these invariant lines.

The proof of this theorem employs geometric singular perturbation theory to demonstrate that there is a transverse intersection of invariant manifolds in system (4.15) in which the traveling wave lies.

Boundedness of the vector field in the C^1 topology

The kinetic terms $\frac{s}{K_S+s}$ and $\frac{a}{K_A+a}$ in f_{bd}, with $K_{S,A} = \delta^\kappa \tilde{K}_{S,A}$ and $\kappa > 0$, are not uniformly bounded in the C^1 topology as $\delta \to 0$. More precisely,

$$\frac{d}{ds}\left(\frac{s}{\delta^\kappa \tilde{K}_S + s}\right) = \frac{\delta^\kappa \tilde{K}_S}{(s + \delta^\kappa \tilde{K}_S)^2} \to \infty \quad \text{for} \quad s \ll \mathcal{O}(\delta^{\kappa/2}).$$

Hence, the perturbation terms in (4.15) are not uniformly bounded in the C^1 topology, and some preparation of the equations is required before geometric singular perturbation theory [20] – [34] can be applied.

We introduce the new dependent variables

$$y = \frac{s}{K_S + s}, \quad w = \frac{a}{K_A + a} \tag{4.16}$$

with inverse coordinate change being given by $s = yK_S/(1-y)$ and $a = wK_A/(1-w)$. This coordinate change compactifies the s and a directions in the phase space in a manner that naturally reflects the component functions of the reaction function. In terms of these new variables, system (4.15) becomes

$$\begin{aligned}
y' &= \frac{v}{K_S}(1-y)^2 \\
v' &= -(cR_d - 1)v + \delta^2 \tilde{a}_1 myw \\
w' &= \frac{r}{K_A}(1-w)^2 \\
r' &= -(c-1)r + \delta^2 \tilde{a}_1 a_2 myw \\
m' &= \delta^2 \frac{\tilde{a}_4}{c}(m-1) - \delta \frac{\tilde{a}_3}{c} myw.
\end{aligned} \tag{4.17}$$

Numerically, the variables v and r remain small along the traveling wave, while the height of the peak in m is large. For $0 < \kappa < 1$, these numerics suggest scaling $v = \delta^{1+\kappa}\tilde{v}$, $r = \delta^{1+\kappa}\tilde{r}$, and $m - 1 = \delta^{\kappa-1}\tilde{m}$ (see remark 4.3.5). Hence, taking into account the scalings of K_S and K_A in (4.14), we see that system (4.17) becomes

$$\begin{aligned}
y' &= \delta \frac{\tilde{v}}{\tilde{K}_S}(1-y)^2 \\
\tilde{v}' &= -(cR_d - 1)\tilde{v} + \tilde{a}_1 \tilde{m} yw + \delta^{1-\kappa} \tilde{a}_1 yw \\
w' &= \delta \frac{\tilde{r}}{\tilde{K}_A}(1-w)^2 \\
\tilde{r}' &= -(c-1)\tilde{r} + \tilde{a}_1 a_2 \tilde{m} yw + \delta^{1-\kappa} \tilde{a}_1 a_2 yw \\
\tilde{m}' &= -\delta \frac{\tilde{a}_3}{c} \tilde{m} yw - \delta^{2-\kappa} \frac{\tilde{a}_3}{c} yw + \delta^2 \frac{\tilde{a}_4}{c} \tilde{m}.
\end{aligned} \tag{4.18}$$

System (4.18) is the fast-slow system that we will use throughout the proof of Theorem 4.3.4. We remark that $\delta \ll \delta^{1-\kappa}$, because we are assuming $0 < \kappa < 1$. This will be crucial in the following analysis.

Geometry of the fast-slow system (4.18)

In system (4.18), $\tilde{v}$ and $\tilde{r}$ are fast variables, while the rest are slow. The reduced slow system is obtained from (4.18) by changing the independent variable to the slow time $\eta = \delta\xi$ and by setting $\delta = 0$,

$$\begin{aligned}
y_\eta &= \frac{\tilde{v}}{\tilde{K}_S}(1-y)^2 \\
0 &= -(cR_d - 1)\tilde{v} + \tilde{a}_1 \tilde{m} y w \\
w_\eta &= \frac{\tilde{r}}{\tilde{K}_A}(1-w)^2 \\
0 &= -(c-1)\tilde{r} + \tilde{a}_1 a_2 \tilde{m} y w \\
\tilde{m}_\eta &= -\frac{\tilde{a}_3}{c}\tilde{m} y w.
\end{aligned} \tag{4.19}$$

The algebraic constraints in (4.19) imply that $\tilde{v} = -\tilde{r} = \frac{\tilde{a}_1}{cR_d - 1}\tilde{m}yw$, where we note that we have used the fact that $a_2(cR_d - 1) = -(c-1)$. Therefore, system (4.19) has an invariant manifold given by

$$\mathcal{M}_0 = \{\tilde{v} = -\tilde{r} = \frac{\tilde{a}_1}{cR_d - 1}\tilde{m}yw\}, \tag{4.20}$$

and in this case, since $\delta = 0$, we label $\mathcal{M}_0$ as a critical manifold.

Setting $\delta = 0$ in system (4.18), we see that the reduced fast dynamics are given by

$$\begin{aligned}
y' &= 0 \\
\tilde{v}' &= -(cR_d - 1)\tilde{v} + \tilde{a}_1 \tilde{m} y w \\
w' &= 0 \\
\tilde{r}' &= -(c-1)\tilde{r} + \tilde{a}_1 a_2 \tilde{m} y w \\
\tilde{m}' &= 0.
\end{aligned} \tag{4.21}$$

Because $cR_d - 1 > 0$ and $c - 1 < 0$, we see that $\tilde{v}$ is decaying exponentially while $\tilde{r}$ is growing exponentially. Hence, $\mathcal{M}_0$ is normally hyperbolic.

Fenichel theory [20], [21], [34] implies that, for sufficiently small δ, the full system (4.18) has a locally invariant slow manifold, $\mathcal{M}_\delta$, which is C^1 $\mathcal{O}(\delta^{1-\kappa})$ close to $\mathcal{M}_0$. In addition, $\mathcal{M}_\delta$ is the graph of a function, which has a regular perturbation expansion, as follows:

$$\begin{aligned}
\tilde{v} &= h_0(\tilde{m}, y, w) + \delta^{1-\kappa} h_1(\tilde{m}, y, w) + \delta h_2(\tilde{m}, y, w) + \text{h.o.t} \\
\tilde{r} &= g_0(\tilde{m}, y, w) + \delta^{1-\kappa} g_1(\tilde{m}, y, w) + \delta g_2(\tilde{m}, y, w) + \text{h.o.t.}
\end{aligned} \tag{4.22}$$

The functions h_i and g_i, $i = 0, 1, 2$, are obtained by computing $d\tilde{v}/d\xi$ and $d\tilde{r}/d\xi$ from the above asymptotic expansion and from system (4.18), respectively, and then by equating these two expressions, which expresses analytically the invariance of $\mathcal{M}_\delta$. By using (4.11), at $\mathcal{O}(1)$, we find

$$h_0 = \frac{\tilde{a}_1}{(cR_d - 1)}\tilde{m}yw,$$
$$g_0 = -h_0. \tag{4.23}$$

At $\mathcal{O}(\delta^{1-\kappa})$, we find

$$h_1 = \frac{\tilde{a}_1}{(cR_d - 1)}yw,$$
$$g_1 = -h_1. \tag{4.24}$$

Finally, at $\mathcal{O}(\delta)$

$$h_2 = -\frac{\tilde{a}_1^2}{(cR_d - 1)^3}yw\tilde{m}\left\{\frac{\tilde{m}w(1-y)^2}{\tilde{K}_S} - \frac{\tilde{m}y(1-w)^2}{\tilde{K}_A} - \frac{\tilde{a}_3(cR_d - 1)}{c\tilde{a}_1}yw\right\},$$
$$g_2 = \frac{1}{a_2}h_2. \tag{4.25}$$

The equations for $\tilde{v}$ and $\tilde{r}$ in (4.21) indicate that if a solution were to leave the slow manifold it would not return and either $\tilde{v}$ or $\tilde{r}$ would tend to infinity as $\xi \to \pm\infty$. Therefore, the entire traveling wave solution must be contained within the slow manifold.

Numerical verification of the asymptotic expansions given in equation (4.22) is shown in figure 4·2. Note the good agreement as expected for $0 < \kappa < 1$.

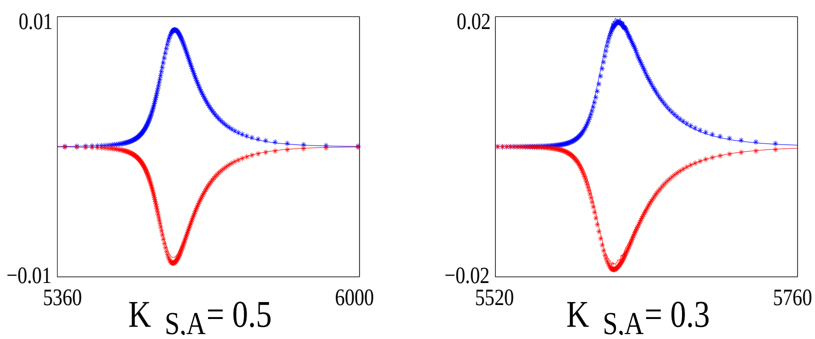

Figure 4·2: A comparison of $\tilde{v}$ (upper curve) and $\tilde{r}$ (lower curve) as computed numerically (*) and using the asymptotic expansion (-) given by equation (4.22), for $K_{S,A} = 0.5$ and $K_{S,A} = 0.3$, and the other parameter values as in equation (4.12).

Remark 4.3.5 *The reason for choosing the scalings* $v = \delta^{1+\kappa}\tilde{v}$, $r = \delta^{1+\kappa}\tilde{r}$, *and* $m - 1 = \delta^{\kappa-1}\tilde{m}$ *can be seen as follows. First, the numerics indicate that the reaction term balances* *with v and r along the wave for $0 < \kappa < 1$, and under these scalings, it is precisely these terms that are $\mathcal{O}(1)$ in the right members of the equations for $\tilde{v}$ and $\tilde{r}$ in (4.17) when the above scalings are employed. Second, y', w' and $\tilde{m}'$ are of the same order in (4.18), as are $\tilde{v}'$ and $\tilde{r}'$. This allows for the reduction to the three dimensional slow manifold $\mathcal{M}_\delta$.*

Dynamics on the slow manifold $\mathcal{M}_\delta$

The dynamics on the slow manifold $\mathcal{M}_\delta$ are obtained by inserting formulas (4.22) into system (4.18) and changing the independent variable to the slow time $\eta = \delta\xi$,

$$\begin{aligned} y_\eta &= \frac{(1-y)^2}{\tilde{K}_S}\left[h_0 + \delta^{1-\kappa}h_1 + \delta h_2 + \mathcal{O}(\delta^{2-\kappa})\right] \\ w_\eta &= \frac{(1-w)^2}{\tilde{K}_A}\left[g_0 + \delta^{1-\kappa}g_1 + \delta g_2 + \mathcal{O}(\delta^{2-\kappa})\right] \\ \tilde{m}_\eta &= -\frac{\tilde{a}_3}{c}\tilde{m}yw - \delta^{1-\kappa}\frac{\tilde{a}_3}{c}yw + \delta\frac{\tilde{a}_4}{c}\tilde{m}. \end{aligned} \tag{4.26}$$

Retaining the $\mathcal{O}(1)$ and $\mathcal{O}(\delta^{1-\kappa})$ terms, we have

$$\begin{aligned} y_\eta &= \frac{\tilde{a}_1}{\tilde{K}_S(cR_d-1)}(1-y)^2\left[\tilde{m}yw + \delta^{1-\kappa}yw\right] \\ w_\eta &= -\frac{\tilde{a}_1}{\tilde{K}_A(cR_d-1)}(1-w)^2\left[\tilde{m}yw + \delta^{1-\kappa}yw\right] \\ \tilde{m}_\eta &= -\frac{\tilde{a}_3}{c}\left[\tilde{m}yw + \delta^{1-\kappa}yw\right]. \end{aligned} \tag{4.27}$$

In turn, from (4.27) we see that up to terms of $\mathcal{O}(\delta)$,

$$\frac{\tilde{K}_S}{(1-y)^2}y_\eta = -\frac{\tilde{K}_A}{(1-w)^2}w_\eta = -\frac{c\tilde{a}_1}{\tilde{a}_3(cR_d-1)}\tilde{m}_\eta. \tag{4.28}$$

Integrating these equalities pairwise, we find that the integral curves of (4.27) are given by

$$\begin{aligned} c_0 &= \frac{\tilde{K}_S}{1-y} + \frac{\tilde{K}_A}{1-w}, \\ \tilde{m} &= \frac{\tilde{a}_3(cR_d-1)}{c\tilde{a}_1}\left(\frac{\tilde{K}_A}{1-w} + c_1\right), \\ \tilde{m} &= \frac{\tilde{a}_3(cR_d-1)}{c\tilde{a}_1}\left(-\frac{\tilde{K}_S}{1-y} + c_2\right), \end{aligned} \tag{4.29}$$

where c_i, $i = 0, 1, 2$ are arbitrary constants that determine which integral curve the solution is on. Moreover, we see that $c_0 + c_1 = c_2$, by equating the two expressions for $\tilde{m}$. Hence, (4.28) determines a two-parameter family of independent integral curves.

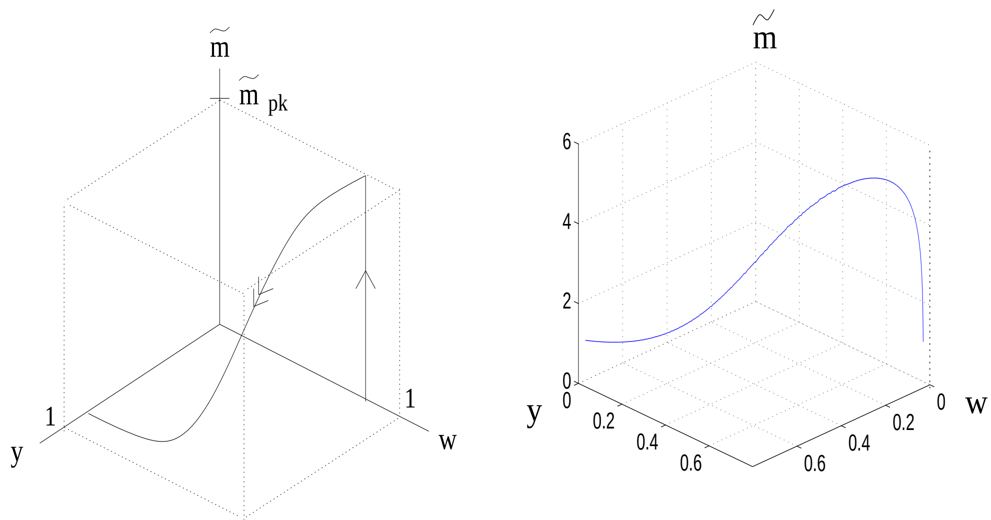

Figure 4·3: On the left is a sketch of an integral curve for system (4.27) near the TW solution. Here $\tilde{m}_{pk}$ is the height of the peak in $\tilde{m}$, as given in equation (4.38). A plot of the numerically computed traveling wave is shown on the right, for $K_{S,A} = 0.3$.

To identify the integral curve corresponding to the traveling wave whose existence we want to establish, we use the full ($\delta > 0$) boundary conditions $y(+\infty) = \frac{1}{1+\delta^\kappa \tilde{K}_S}$ and $w(+\infty) = 0$, as well as $y(-\infty) = 0$ and $w(-\infty) = \frac{1}{1+\delta^\kappa \tilde{K}_A}$. The integral curve which satisfies these boundary conditions is defined by $\delta^\kappa c_0 = 1 + \delta^\kappa \tilde{K}_S + \delta^\kappa \tilde{K}_A$. In other words, in terms of the y and w variables, the integral curve $\mathcal{I}$, which constitutes part of the traveling wave, is given up to $\mathcal{O}(\delta)$ by

$$\frac{\delta^\kappa \tilde{K}_S}{1-y} + \frac{\delta^\kappa \tilde{K}_A}{1-w} = 1 + \delta^\kappa \tilde{K}_S + \delta^\kappa \tilde{K}_A. \tag{4.30}$$

Similarly, in y and $\tilde{m}$, this integral curve $\mathcal{I}$ contains the point $(\frac{1}{1+\delta^\kappa \tilde{K}_S}, 0)$ and is given by $\delta^\kappa c_2 = 1 + \delta^\kappa \tilde{K}_S$. Also, we have $\delta^\kappa c_1 = -\delta^\kappa \tilde{K}_A$, since $c_0 + c_1 = c_2$. Therefore, along the integral curve $\mathcal{I}$, $\tilde{m}$ is given as a function of w, respectively y, by

$$\begin{aligned} \delta^\kappa \tilde{m} &= \frac{\tilde{a}_3(cR_d - 1)}{c\tilde{a}_1}\left(\frac{\delta^\kappa \tilde{K}_A}{1-w} - \delta^\kappa \tilde{K}_A\right), \\ \delta^\kappa \tilde{m} &= \frac{\tilde{a}_3(cR_d - 1)}{c\tilde{a}_1}\left(-\frac{\delta^\kappa \tilde{K}_S}{1-y} + 1 + \delta^\kappa \tilde{K}_S\right). \end{aligned} \tag{4.31}$$

To complete the construction of the traveling wave up to $\mathcal{O}(\delta)$, we append to the integral curve $\mathcal{I}$ the line segment $\mathcal{L}^- = \{(y, w, \tilde{m}) | y = 0, w = \frac{1}{1+\delta^\kappa \breve{K}_A}, \tilde{m} \in [0, \tilde{m}_{pk}]\}$, where $\tilde{m}_{pk}$ is a constant which will be determined explicitly in section 4.3.3. The union

$$\mathcal{S}_\delta = \mathcal{I} \bigcup \mathcal{L}^- \tag{4.32}$$

is the traveling wave to leading order.

Remark 4.3.6 *The reason for including the $\mathcal{O}(\delta^{1-\kappa})$ terms when determining the leading order traveling wave is that the integral curves (4.29) become degenerate as $\delta \to 0$.*

Completion of the proof of theorem 4.3.4

In this section, we complete the proof of Theorem 4.3.4 by identifying certain submanifolds of the slow manifold $\mathcal{M}_\delta$ and by showing that these submanifolds intersect transversely along a one-dimensional curve that is given by the set $\mathcal{S}_\delta$, recall (4.32), up to $\mathcal{O}(\delta)$. The traveling wave will be the heteroclinic orbit that lies in this transverse intersection.

First, notice that for the full system (4.18), the manifold $\mathcal{N}_\delta \equiv \mathcal{N}_\delta^- \bigcup \mathcal{N}_\delta^+ = \{y = \tilde{v} = \tilde{r} = 0\} \bigcup \{w = \tilde{r} = \tilde{v} = 0\}$ is invariant for all $\delta \geq 0$. On this manifold, the dynamics of $\tilde{m}$ are given by $\tilde{m}_\xi = \delta^2 \frac{\tilde{a}_4}{c} \tilde{m}$, while y, $\tilde{v}$, w, and $\tilde{r}$ are constant. These dynamics correspond to the behavior of the microorganisms in the absence of any reaction and occur on the "super-slow" timescale which is $\mathcal{O}(\delta^2)$. This suggests that we carry out a fast-slow decomposition of the dynamics within $\mathcal{M}_\delta$.

In fact, $\mathcal{N}_\delta$ is a super-slow manifold for (4.26). To see this, rewrite the system in terms of the variable $\zeta = \delta\eta$ so that the derivatives balance with the $\mathcal{O}(\delta)$ terms on the right hand sides. In order to observe these super-slow dynamics, we must have $h_0 + \delta^{1-\kappa} h_1 = 0$. This is true if either $y = 0$ or $w = 0$. Note that these two conditions both imply that $h_2 = g_2 = 0$. Therefore, on $\mathcal{N}_\delta$ the dynamics are given, to leading order, by

$$\begin{aligned} y_\zeta &= 0 \\ w_\zeta &= 0 \\ \tilde{m}_\zeta &= \frac{\tilde{a}_4}{c} \tilde{m}. \end{aligned} \tag{4.33}$$

We see that on $\mathcal{N}_\delta^-$, the lines for fixed w are invariant, and $\tilde{m}$ grows exponentially away from 0. Similarly, on $\mathcal{N}_\delta^+$ the lines for fixed y are invariant and $\tilde{m}$ grows exponentially away from 0.

Consider the line segments

$$\begin{aligned}\mathcal{L}^- &= \{(0, w_-, \tilde{m}) : w_-\text{fixed}, \quad \tilde{m} \in [0, \tilde{m}_{pk}]\},\\ \mathcal{L}^+ &= \{(y_+, 0, 0) : y_+ \in \left(\frac{1}{1+\delta^\kappa \tilde{K}_S} - \epsilon, \frac{1}{1+\delta^\kappa \tilde{K}_S} + \epsilon\right)\},\end{aligned} \tag{4.34}$$

for some $\epsilon > 0$ and a constant $\tilde{m}_{pk}$ which will be defined in section 4.3.3 (see figure 4·4). Using the integral curves given in (4.29) we will track $\mathcal{L}^-$ forward and $\mathcal{L}^+$ backward to the plane $\{y = w\}$ and show they intersect transversely in this plane.

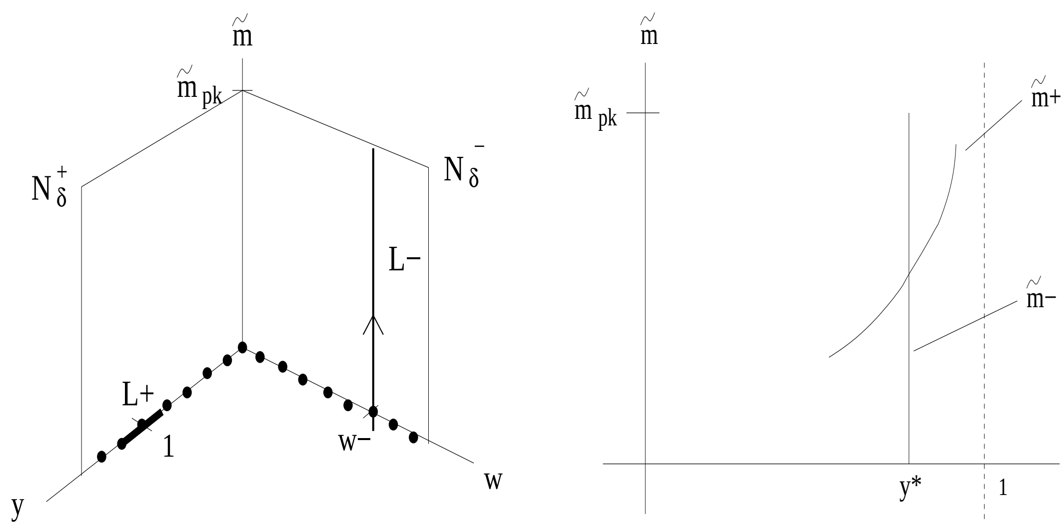

Figure 4·4: On the left is a schematic diagram of the phase space of (4.33), showing the slow manifold $\mathcal{N}_0$ and the lines $\mathcal{L}^\pm$. A sketch of the functions $\tilde{m}^-(y)$ and $\tilde{m}^+(y)$ in the plane $\{y = w\}$ showing the transverse intersection is shown on the right, where $y* = \frac{\tilde{K}_A w_-}{\tilde{K}_A + \tilde{K}_S - \tilde{K}_S w_-}$.

First, we will track $\mathcal{L}^-$ forward. Any integral curve which intersects $\mathcal{L}^-$ must contain a point of the form $(0, w_-, \tilde{m})$. Using this information we can determine the constants c_0 and c_1 for the integral curves which determine the evolution of $\mathcal{L}^-$. We find that $c_0 = \tilde{K}_S + \frac{\tilde{K}_A}{1-w_-}$ and $c_1 = \frac{c\tilde{a}_1}{\tilde{a}_3(cR_d - 1)}\tilde{m} - \frac{\tilde{K}_A}{(1-w_-)}$. This implies that in the plane $\{y = w\}$, by (4.29), we have

$$y = w = \frac{\tilde{K}_A w_-}{\tilde{K}_A + \tilde{K}_S - \tilde{K}_S w_-}; \quad \tilde{m}^- \in [0, \tilde{m}_{pk}]. \tag{4.35}$$

We can use a similar procedure to track $\mathcal{L}^+$ backward. Any integral curve intersecting $\mathcal{L}^+$ must contain a point of the form $(y_+, 0, 0)$, which implies that $c_0 = \tilde{K}_A + \frac{\tilde{K}_S}{1-y_+}$ and

$c_2 = \frac{\tilde{K}_S}{(1-y_+)}$. Hence, on the plane $\{y = w\}$ we have

$$\tilde{m}^+(y) = \frac{\tilde{K}_A \tilde{a}_3 (cR_d - 1)}{\tilde{a}_1 c} \left(\frac{y}{1-y}\right); \quad y \in \left(\frac{1}{1+\delta^\kappa \tilde{K}_S} - \epsilon, \frac{1}{1+\delta^\kappa \tilde{K}_S} + \epsilon\right). \tag{4.36}$$

Graphs of $\tilde{m}^-(y)$ and $\tilde{m}^+(y)$, which intersect transversely, are shown in figure 4·4. Because the images of $\mathcal{L}^-$ and $\mathcal{L}^+$ intersect transversely, we know that a trajectory connecting the point $(0, w_-, 0)$ and the line $\mathcal{L}^+$ will persist under the addition of the higher order terms [55].

All that remains to be shown is that if we chose $w_- = \frac{1}{1+\delta^\kappa \tilde{K}_A}$, then $y_+ = \frac{1}{1+\delta^\kappa \tilde{K}_S}$. This will result from the following argument. Notice that, using (4.17),

$$a_2[v' + (cR_d - 1)v] = r' + (c-1)r.$$

If we use the fact that $y(+\infty) = y_+$, $w(-\infty) = w_-$, and $y(-\infty) = w(+\infty) = 0$, and integrate from $-\infty$ to $+\infty$, we see that

$$a_2(cR_d - 1)\frac{\delta^\kappa \tilde{K}_S y_+}{1-y_+} = -(c-1)\frac{\delta^\kappa \tilde{K}_A w_-}{1-w_-}, \tag{4.37}$$

and using the fact that $a_2(cR_d - 1) = -(c-1)$ we see that $y_+ = \frac{\tilde{K}_A w_-}{\tilde{K}_S + (\tilde{K}_A - \tilde{K}_S)w_-}$. Therefore, if we chose $w_- = \frac{1}{1+\delta^\kappa \tilde{K}_A}$, then we must also have $y_+ = \frac{1}{1+\delta^\kappa \tilde{K}_S}$. The reason for this is that the boundary conditions are encoded in the wave speed.

The proof of Theorem 4.3.4 is completed by rewriting the expression for $\mathcal{I}$ given in equations (4.30) and (4.31) in terms of the variables s, a, and m.

Remark 4.3.7 *We briefly comment on the choice of the lines $\mathcal{L}^\pm$. Because $\mathcal{L}^-$ is the unstable manifold of the point $(0, w^-, 0)$ within $\mathcal{N}_0^-$, it is natural to track its forward evolution. Since we are interested in a solution that is asymptotic to a point of the form $(y^+, 0, 0)$, one might initially attempt to track its stable manifold backward. However, this would then require the transverse intersection of a two-dimensional manifold with a one-dimensional manifold in a three-dimensional phase space, which is, in general, not generic. Thus, we track the stable manifold of the line $\mathcal{L}^+$ backward, so that both tracked manifolds have dimension two and their intersection is one-dimensional.*

4.3.3 The dependence of the peak height on K_S and K_A

In this section, we compute the peak height of the $\tilde{m}$ component of the traveling wave first to leading order and then also up to and including the first-order corrections. First, we determine the leading order value using equation (4.31). The maximum value of $\tilde{m}$ will be attained when $y = 0$, or equivalently when $w = \frac{1}{1+\delta^\kappa \tilde{K}_A}$. Thus, we see that the peak height is defined by

$$\delta^\kappa \tilde{m}_{pk} \equiv \frac{\tilde{a}_3(cR_d - 1)}{c\tilde{a}_1} = \frac{\tilde{a}_3(R_d - 1)}{\tilde{a}_1(a_2 + 1)}, \tag{4.38}$$

where we have used the value of c given in (4.11).

The value of $\delta^\kappa \tilde{m}_{pk}$ to leading order, $\frac{\tilde{a}_3(R_d-1)}{\tilde{a}_1(a_2+1)}$, is exactly the upper bound on the peak height obtained in [40]. In other words, we have found that their bound is sharp, up to higher order effects in δ. Note that the bound in [40] is given in terms of the dimensional parameters (see [43] for the relationship between the dimensional and nondimensional parameters).

Remark 4.3.8 *In the limit as $\delta \to 0$, $\tilde{m}_{pk} \to \infty$. This is due to the degenerate nature of the integral curves in the singular limit. If we were to define a new variable $\hat{m} = \delta^\kappa \tilde{m}$, then $\hat{m}_{pk}$ would remain $\mathcal{O}(1)$ as $\delta \to 0$, and this would suggest working with the variable $\hat{m}$ throughout the analysis. However, this prevents one from balancing the derivatives of y and w with that of $\hat{m}$ on the slow manifold $\mathcal{M}_\delta$. Therefore, we work with the variable $\tilde{m}$ instead.*

Numerically, one observes that the height of the peak in the $\tilde{m}$ component increases as the half saturation constants K_S and K_A decrease (see figure 4·5), as long as one remains in the regime where the traveling wave is stable. This increase is confirmed analytically using the analysis on the slow manifold $\mathcal{M}_\delta$, as we show now.

We compute $d\tilde{m}/dy$ using (4.26) and the definitions of h_i, and g_i, $i = 0, 1, 2$ given in (4.23), (4.24), and (4.25),

$$\frac{d\tilde{m}}{dy} = -\frac{\tilde{m}_{pk}\tilde{K}_S}{(1-y)^2}\left(\frac{1 - \delta f_1(y, w, \tilde{m})}{1 - \delta f_2(y, w, \tilde{m})}\right), \tag{4.39}$$

where f_1 and f_2 are given by

$$\begin{aligned} f_1(y, w, \tilde{m}) &= \frac{\tilde{a}_4}{\tilde{a}_3}\left(\frac{\tilde{m}}{\tilde{m}yw + \delta^{1-\kappa}yw}\right) \\ f_2(y, w, \tilde{m}) &= \frac{\tilde{a}_1}{(cR_d - 1)^2}\left(\frac{\tilde{m}}{\tilde{m} + \delta^{1-\kappa}}\right)\left(\frac{\tilde{m}w(1-y)^2}{\tilde{K}_S} - \frac{\tilde{m}y(1-w)^2}{\tilde{K}_A} - \frac{\tilde{a}_3(cR_d - 1)}{c\tilde{a}_1}yw\right). \end{aligned} \tag{4.40}$$

In order that equation given (4.39) depends only on $\tilde{m}$ and y, we eliminate w using the

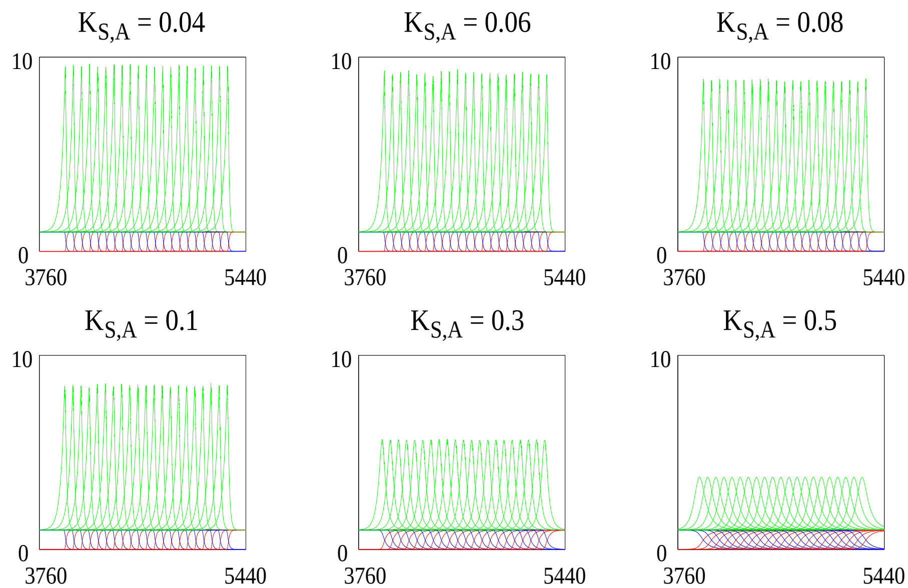

Figure 4·5: Plot of the traveling wave, show at successive time steps, for increasing values of the half saturation constants K_S and K_A. Note that as the half saturation increase, the height of the peak decreases.

leading order integral curve $w = w(y)$ given in (4.30). That is,

$$w = \frac{(1 + \delta^\kappa \tilde{K}_S)(1 - y) - \delta^\kappa \tilde{K}_S}{(1 + \delta^\kappa \tilde{K}_S + \delta^\kappa \tilde{K}_A)(1 - y) - \delta^\kappa \tilde{K}_S}.$$

Inserting the above expression into the functions f_1 and f_2, we obtain the leading order differential equation for $\tilde{m}$ in terms of y. Integrating this equation numerically to determine the height of the peak in $\tilde{m}$, we see (figure 4·6) that m_{pk} decreases as K_S and K_A increase. In figure 4·6, we also see that the analytical results agree with the numerical results, except for some higher-order corrections. This provides further evidence that, for $0 < \kappa < 1$, the entire traveling wave solution really is contained on the three-dimensional slow manifold $\mathcal{M}_\delta$.

4.3.4 Bifurcation to Periodic Waves

As previously mentioned, the geometry of system (4.17) changes when $\kappa > 1$. Most prominently, as κ increases (or K_S and K_A decrease) the traveling wave loses stability to a periodic wave. In this section, we demonstrate this behavior by numerically investigating the bioremediation model.

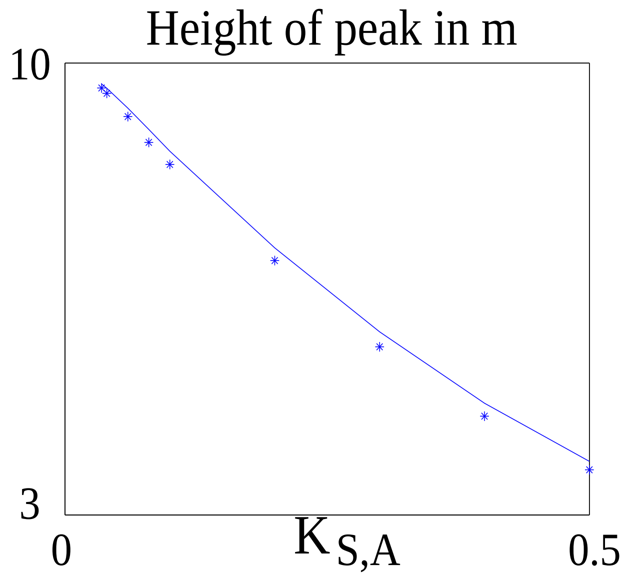

Figure 4·6: Graph of the height of the peak in m versus the half saturation constants K_S and K_A (where $K_S = K_A$) as computed numerically (*) and as computed using the higher order corrections on the slow manifold $\mathcal{M}(\delta)$ (solid curve), via equation (4.39).

Numerical Methods

In order to integrate system (4.4), we have used a moving grid code which is described in detail in [4]. Because numerical integration must be performed on a finite domain, the simulations have been run for $x \in [0, 10000]$, and the asymptotic conditions given in (4.10) have been used as boundary conditions at the endpoints of the spatial interval. This domain is sufficiently large so that the effects of the finite domain are exponentially small in δ over all of the intervals of time of the simulations. The initial data used are as follows. The microorganism concentration, M, was taken to be constant and equal to 1, the substrate concentration, S, was taken to be 1 everywhere except near the left edge, where it linearly decreases to 0, and the acceptor concentration, A, was taken to be 0 everywhere except near the left edge, where it linearly increases to 1.

For all results presented in this section, the parameter values used are

$$a_1 = 0.011 \quad a_2 = 0.3450 \quad a_3 = 0.0885 \quad a_4 = 0.0218 \quad R_d = 3.0, \tag{4.41}$$

which are the same as those given in section 4.3.1. We remark, however, that the bioremediation model has been numerically integrated for other values of these parameters, to ensure that the traveling wave is not unstable relative to small changes in them.

We are interested in the behavior of system (4.4) for a range of values of the half saturation constants. In particular, we have run numerical simulations for $K_S = K_A \in [0.01, 1]$. For this paper we have taken $K_S = K_A$ to simplify the scope of the numerical simulations.

Geometry of the Phase Space

Based upon the preceding existence construction, we see that for $0 < \kappa < 1$, or $0.1 < K_{S,A} < 1$, the entire traveling wave is contained within a three-dimensional slow manifold within the five-dimensional phase space, given by the asymptotic expansions (4.22). This was verified numerically in figure 4·2.

Similarly, we can see numerically that this is not the case when $\kappa > 1$, or $K_{S,A} < 0.1$, given the values of the other parameters. In other words, if we plot the asymptotic expansion in (4.22) for $\kappa > 1$ (evaluated using the numerical values of y, w, and $\tilde{m}$) against the values of $\tilde{v}$ and $\tilde{r}$ as computed numerically, we see that they do not agree (see figure 4·7). This indicates that the slow manifold $\mathcal{M}_\delta$ no longer contains the traveling wave solution for these parameter values. Because the bifurcation to a periodic wave happens for small

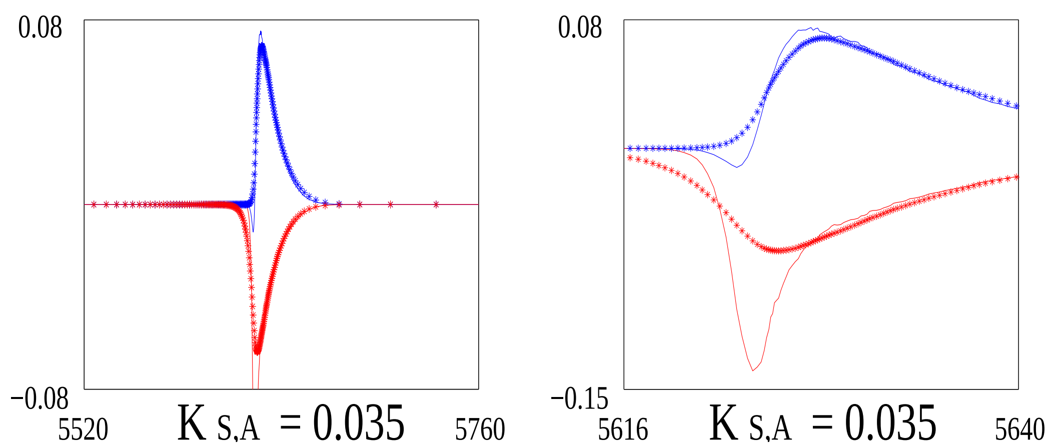

Figure 4·7: A comparison of $\tilde{v}$ and $\tilde{r}$ as computed numerically (*) and using the asymptotic expansion (-), for $K_S = K_A = 0.035$. The figure on the right is a close-up of that on the left.

values of K_S and K_A, understanding this change in geometry may provide insight into why the traveling wave loses stability.

Periodic Waves

As K_S and K_A decrease further, the traveling wave loses stability to a periodic wave. This bifurcation happens for $K_S = K_A \approx 0.032$, which corresponds to $\kappa \approx 3/2$, and is shown in figure 4·8. Each graph shows snapshots of the traveling wave at time intervals of ten units. For values of $K_{S,A}$ below the bifurcation value, the peak height in the m component of the wave varies periodically as the wave travels. Notice that the frequency of oscillation appears to be constant. This suggests that the traveling wave loses stability as the result of a Hopf bifurcation, where two conjugate eigenvalues cross the imaginary axis.

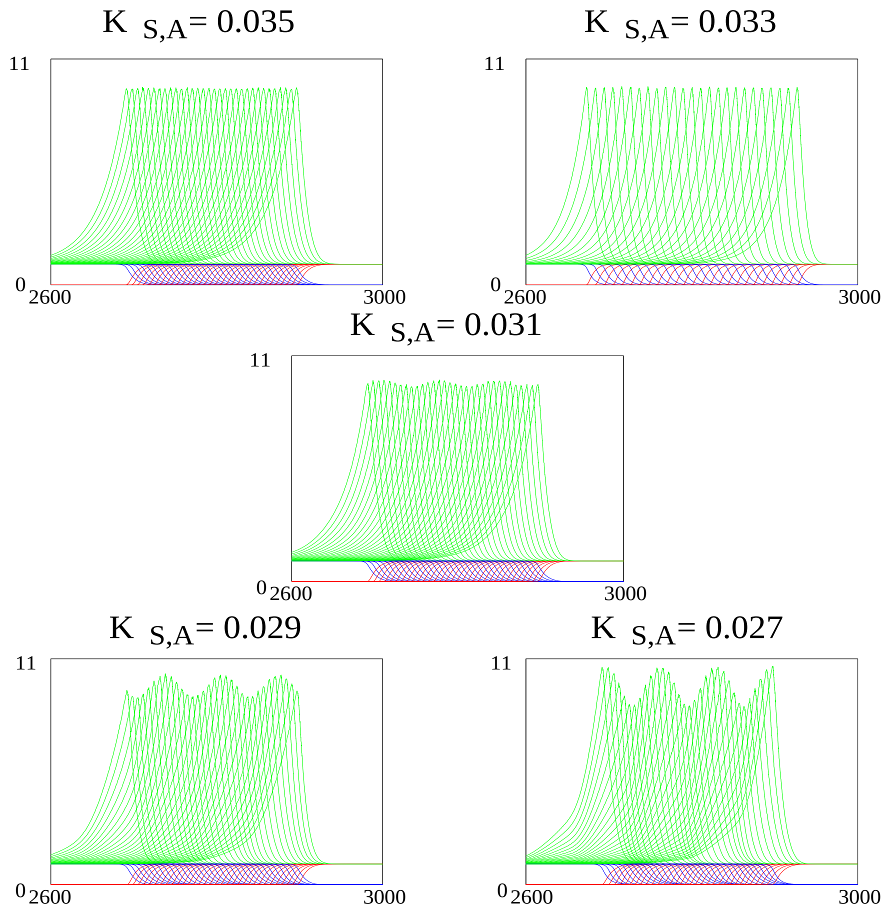

Figure 4·8: Each frame shows 33 snapshots of the numerically observed traveling wave at intervals of 10 time units. Note that the traveling wave appears to lose stability to a time-periodic wave approximately for a value of $K_{S,A}$ somewhere inside $(0.031, 0.033)$.

If we approximate the period of oscillation using the numerical results shown in figure 4·8, we find that the period is approximately 102.7 nondimensional time units. Consequently, this implies the frequency of oscillation is about $2\pi/102.7 \approx 0.061$. Therefore, we expect that the stability analysis will show that two conjugate eigenvalues cross the imaginary axis with imaginary part near 0.061 as $K_{S,A}$ decrease through 0.032.

In order to verify this numerically observed behavior, stability analysis for the parameter regime in which we have constructed the traveling wave, $0 < \kappa < 1$, must be performed. This will be done in section 4.4 below. In addition, construction and stability analysis of the traveling wave in the regime $\kappa > 1$ needs to be carried out in order to investigate the bifurcation. This is the subject of future work.

4.4 Linear Stability of the Wave: Locating the Spectrum

In the previous section, the traveling wave for the bioremediation model was constructed for sufficiently large values of the half saturation constants K_S and K_A. We now use this result to determine the linear stability properties of the wave.

Recall that the bioremediation model we are studying is

$$\begin{aligned} R_d \frac{\partial S}{\partial t} - \frac{\partial^2 S}{\partial x^2} + \frac{\partial S}{\partial x} &= -a_1 f_{bd} \\ \frac{\partial A}{\partial t} - \frac{\partial^2 A}{\partial x^2} + \frac{\partial A}{\partial x} &= -a_1 a_2 f_{bd} \\ \frac{\partial M}{\partial t} &= a_3 f_{bd} - a_4(M-1) \\ f_{bd} &= M\left(\frac{S}{K_S + S}\right)\left(\frac{A}{K_A + A}\right), \end{aligned} \tag{4.42}$$

where $x \in \mathbb{R}$ and $t > 0$.

We constructed the traveling wave using the variables (Y, W, M), where $Y = \frac{S}{K_S+S}$ and $W = \frac{A}{K_A+A}$, and the moving coordinate $\xi = x - ct$. In addition, the parameters were scaled as $a_1 = \delta^2 \tilde{a}_1$, $a_2 = a_2$, $a_3 = \delta \tilde{a}_3$, $a_4 = \delta^2 \tilde{a}_4$, $K_S = \delta^\kappa \tilde{K}_S$ and $K_A = \delta^\kappa \tilde{K}_A$. (See equations (4.13) and (4.14).) In terms of these variables, the governing PDE is

$$\begin{aligned} R_d Y_t &= Y_{\xi\xi} + (cR_d - 1)Y_\xi + \frac{2Y_\xi^2}{1-Y} - \delta^{2-\kappa}\frac{\tilde{a}_1}{\tilde{K}_S}(1-Y)^2 YWM \\ W_t &= W_{\xi\xi} + (c-1)W_\xi + \frac{2W_\xi^2}{1-W} - \delta^{2-\kappa}\frac{\tilde{a}_1 a_2}{\tilde{K}_A}(1-W)^2 YWM \\ M_t &= cM_\xi - \delta^2 \tilde{a}_4(M-1) + \delta \tilde{a}_3 YWM. \end{aligned} \tag{4.43}$$

We denote the traveling wave solution, constructed in the previous section, by (y_{tw}, w_{tw}, m_{tw}). Let

$$\begin{pmatrix} Y \\ W \\ M \end{pmatrix} = \begin{pmatrix} y_{tw} \\ w_{tw} \\ m_{tw} \end{pmatrix} + \begin{pmatrix} p \\ q \\ u \end{pmatrix},$$

substitute into equation (4.43), and retain only the linear terms in $(p, q, u)^t$. The equation describing the evolution of (p, q, u) is

$$\begin{pmatrix} p_t \\ q_t \\ u_t \end{pmatrix} = D \begin{pmatrix} p_{\xi\xi} \\ q_{\xi\xi} \\ u_{\xi\xi} \end{pmatrix} + M(\xi) \begin{pmatrix} p_\xi \\ q_\xi \\ u_\xi \end{pmatrix} + N(\xi) \begin{pmatrix} p \\ q \\ u \end{pmatrix}, \tag{4.44}$$

where

$$D = \begin{pmatrix} \frac{1}{R_d} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad M(\xi) = \begin{pmatrix} \frac{1}{R_d}(cR_d - 1) + \frac{1}{R_d}\frac{4y'}{1-y} & 0 & 0 \\ 0 & (c-1) + \frac{4w'}{1-w} & 0 \\ 0 & 0 & c \end{pmatrix}, \qquad (4.45)$$

$$N(\xi) = \begin{pmatrix} \frac{2(y')^2}{R_d(1-y)^2} - \delta^{2-\kappa}\frac{\tilde{a}_1}{R_d\tilde{K}_S}F(\xi) & -\delta^{2-\kappa}\frac{\tilde{a}_1}{R_d\tilde{K}_S}(1-y)^2ym & -\delta^{2-\kappa}\frac{\tilde{a}_1}{R_d\tilde{K}_S}(1-y)^2yw \\ -\delta^{2-\kappa}\frac{\tilde{a}_1a_2}{\tilde{K}_A}(1-w)^2wm & \frac{2(w')^2}{(1-w)^2} - \delta^{2-\kappa}\frac{\tilde{a}_1a_2}{\tilde{K}_A}G(\xi) & -\delta^{2-\kappa}\frac{\tilde{a}_1a_2}{\tilde{K}_A}(1-w)^2yw \\ \delta\tilde{a}_3wm & \delta\tilde{a}_3ym & -\delta^2\tilde{a}_4 + \delta\tilde{a}_3yw \end{pmatrix}, \qquad (4.46)$$

and we have used the fact that

$$\frac{2(s_0+s)^2}{1-(z_0+z)} = \frac{2s_0^2}{1-z_0} + \frac{4s_0}{1-z_0}s + \frac{2s_0^2}{(1-z_0)^2}z + \mathcal{O}(s^2, z^2, sz).$$

Also in the above equation, $y = y_{tw}$, $w = w_{tw}$, $m = m_{tw}$, and

$$\begin{aligned} F(\xi) &= (1-y_{tw})^2 w_{tw}m_{tw} - 2(1-y_{tw})y_{tw}w_{tw}m_{tw} \\ G(\xi) &= (1-w_{tw})^2 y_{tw}m_{tw} - 2(1-w_{tw})y_{tw}w_{tw}m_{tw}. \end{aligned} \qquad (4.47)$$

We will denote the linear operator in (4.44) by

$$\mathcal{L} = D\partial_\xi^2 + M(\xi)\partial_\xi + N(\xi). \qquad (4.48)$$

The spectrum of this operator determines the linearized stability of the traveling wave. In order to compute the spectrum, we divide it into two pieces: the point spectrum, denoted by $\sigma_{pt}(\mathcal{L})$, which consists of all isolated eigenvalues of finite multiplicity, and the essential spectrum, denoted by $\sigma_{ess}(\mathcal{L})$, which is the complement of the point spectrum: $\sigma_{ess}(\mathcal{L}) = \sigma(\mathcal{L})\backslash\sigma_{pt}(\mathcal{L})$.

4.4.1 Essential Spectrum

In order to determine the essential spectrum, we use theorem 2.1.3. We remark that, although the theorem is stated for scalar equations only, it can also be applied to systems [31]. In addition, the fact that the matrix D defined above is not invertible will not affect the result [33]. Thus, the essential spectrum of $\mathcal{L}$ is determined by its asymptotic limits at $\pm\infty$, $\mathcal{L}^\pm \equiv D\partial_\xi^2 + M^\pm\partial_\xi + N^\pm$, where $M^\pm$ and $N^\pm$ are the limits of M and N as $\xi \to \pm\infty$, respectively. Using the fact that

$$\begin{aligned}
y_{tw}(-\infty) &= 0 & y_{tw}(+\infty) &= \frac{1}{1+\delta^\kappa \tilde{K}_S} \\
w_{tw}(-\infty) &= \frac{1}{1+\delta^\kappa \tilde{K}_A} & w_{tw}(+\infty) &= 0 \\
m_{tw}(-\infty) &= 1 & m_{tw}(+\infty) &= 1 \\
y'_{tw}(\pm\infty) &= 0 & w'_{tw}(\pm\infty) &= 0,
\end{aligned} \tag{4.49}$$

we find that the "parabolas" which determine the location of the essential spectrum at $-\infty$ are

$$\begin{aligned}
\lambda_1^- &= -\delta^2 \tilde{a}_4 + ick \\
\lambda_2^- &= -k^2 + i(c-1)k \\
\lambda_3^- &= -\frac{1}{R_d}k^2 - \delta^{2-\kappa}\frac{\tilde{a}_1}{R_d \tilde{K}_S(1+\delta^\kappa \tilde{K}_A)} + i\frac{cR_d - 1}{R_d}k,
\end{aligned} \tag{4.50}$$

and those at $+\infty$ are

$$\begin{aligned}
\lambda_1^+ &= -\delta^2 \tilde{a}_4 + ick \\
\lambda_2^+ &= -k^2 - \delta^{2-\kappa}\frac{\tilde{a}_1 a_2}{\tilde{K}_A(1+\delta^\kappa \tilde{K}_S)} + i(c-1)k \\
\lambda_3^+ &= -\frac{1}{R_d}k^2 + i\frac{(cR_d - 1)}{R_d}k,
\end{aligned} \tag{4.51}$$

where $k \in \mathbb{R}$ (see figure 4·9). Note that the parabolas given by λ_2^- and λ_3^+ both touch the imaginary axis at the origin. In addition, in the limit $\delta \to 0$, the parabolas λ_2^+ and λ_3^- also touch the imaginary axis at the origin, and the curves $\lambda_1^\pm$ coincide with the imaginary axis.

4.4.2 Point Spectrum

In order to determine the location of the point spectrum, we construct the Evans function associated with equation (4.44). We begin by writing the eigenvalue problem associated with (4.44) as a system of first order ODEs,

$$\frac{dU}{d\xi} = A(\xi, \lambda)U, \tag{4.52}$$

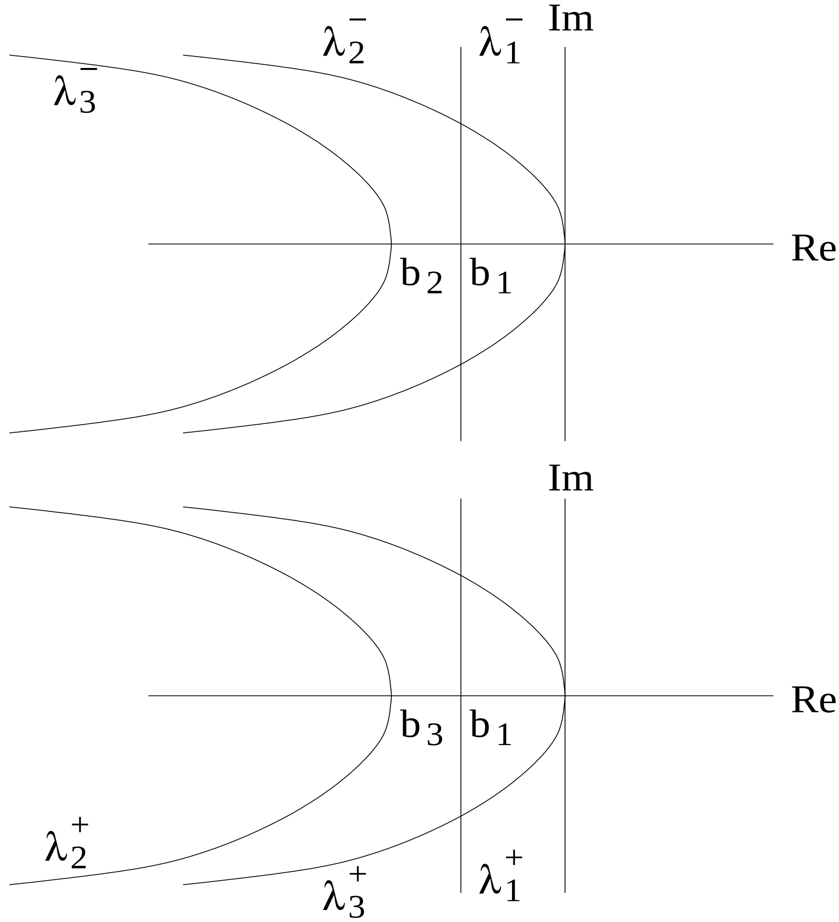

Figure 4·9: The parabolas determining the location of the essential spectrum of $\mathcal{L}$. Here $b_1 = -\delta^2\tilde{a}_4$, $b_2 = -\delta^{2-\kappa}\frac{\tilde{a}_1}{R_d\tilde{K}_S(1+\delta^\kappa\tilde{K}_A)}$, and $b_3 = -\delta^{2-\kappa}\frac{\tilde{a}_1 a_2}{\tilde{K}_A(1+\delta^\kappa\tilde{K}_S)}$.

where $U = (p, b, q, d, u)^T$, and b and d are defined by the following equations:

$$p' = -\delta^{-\kappa}\frac{2v_{tw}(1-y_{tw})}{\tilde{K}_S}p + \delta^{-\kappa}\frac{(1-y_{tw})^2}{\tilde{K}_S}b$$
$$q' = -\delta^{-\kappa}\frac{2r_{tw}(1-w_{tw})}{\tilde{K}_A}q + \delta^{-\kappa}\frac{(1-w_{tw})^2}{\tilde{K}_A}d. \tag{4.53}$$

Remark 4.4.1 *The choice of b and d was determined as follows. Recall that the traveling wave was constructed in a scaled version of the variables (y, v, w, r, m), where $v = s_\xi$ and $r = a_\xi$. However, because of the fact that $v_{tw} \neq y'_{tw}$ and $r_{tw} \neq w'_{tw}$, it does not seem optimal to choose $p' = b$ and $q' = d$. Instead, the above choice of b and d results from computing the variational equations for y and w.*

The fact that $s_{tw} = \frac{K_S y_{tw}}{1-y_{tw}}$ implies $v_{tw} = s'_{tw} = \frac{K_S y'_{tw}}{(1-y_{tw})^2}$. Similarly, $r_{tw} = a'_{tw} = \frac{K_A w'_{tw}}{(1-w_{tw})^2}$. Also recall that, in the existence construction (see the paragraph preceding equation (4.18)), we scaled the dependent variables as $y_{tw} = y_{tw}$, $v_{tw} = \delta^{1+\kappa}\tilde{v}_{tw}$, $w_{tw} = w_{tw}$, $r_{tw} = \delta^{1+\kappa}\tilde{r}_{tw}$, and $m_{tw} = \delta^{\kappa-1}\tilde{m}_{tw} + 1$. Using the fact that

$$0 = y''_{tw} + (cR_d - 1)y'_{tw} + \frac{2(y'_{tw})^2}{1-y_{tw}} - \delta^{2-\kappa}\frac{\tilde{a}_1}{\tilde{K}_S}(1-y_{tw})^2 y_{tw} w_{tw} m_{tw}$$
$$0 = w''_{tw} + (c-1)w'_{tw} + \frac{2(w'_{tw})^2}{1-w_{tw}} - \delta^{2-\kappa}\frac{\tilde{a}_1 a_2}{\tilde{K}_A}(1-w_{tw})^2 y_{tw} w_{tw} m_{tw},$$

we can write the second order eigenvalue problem, $\lambda(p,q,u)^T = \mathcal{L}(\xi)(p,q,u)^T$, as an ODE of the form (4.52), where

$$A(\xi,\lambda) = \begin{pmatrix} -\delta\frac{2\tilde{v}(1-y)}{\tilde{K}_S} & \delta^{-\kappa}\frac{(1-y)^2}{\tilde{K}_S} & 0 & 0 & 0 \\ F_1 & -(cR_d-1) & \delta^{1+\kappa}\tilde{a}_1 yH & 0 & \delta^2\tilde{a}_1 yw \\ 0 & 0 & -\delta\frac{2\tilde{r}(1-w)}{\tilde{K}_A} & \delta^{-\kappa}\frac{(1-w)^2}{\tilde{K}_A} & 0 \\ \delta^{1+\kappa}\tilde{a}_1 a_2 wH & 0 & F_2 & -(c-1) & \delta^2\tilde{a}_1 a_2 yw \\ -\delta^{\kappa}\frac{\tilde{a}_3}{c}wH & 0 & -\delta^{\kappa}\frac{\tilde{a}_3}{c}yH & 0 & \frac{\lambda+\delta^2\tilde{a}_4}{c} - \delta\frac{\tilde{a}_3}{c}yw \end{pmatrix}, \tag{4.54}$$

and

$$H = H(\tilde{m}) = (\tilde{m} + \delta^{1-\kappa})$$
$$F_1 = F_1(y,\tilde{m}) = \delta^{\kappa}\frac{\lambda R_d \tilde{K}_S}{(1-y)^2} + \delta^{1+\kappa}\tilde{a}_1 H$$
$$F_2 = F_2(w,\tilde{m}) = \delta^{\kappa}\frac{\lambda \tilde{K}_A}{(1-w)^2} + \delta^{1+\kappa}\tilde{a}_1 a_2 yH.$$

For ease of notation in the above equation $y = y_{tw}$, $w = w_{tw}$, $m = m_{tw}$, and we will continue to use this notation in equations below.

Next, we scale b, d, u, and λ as $b = \delta^{1+\kappa}\tilde{b}$, $d = \delta^{1+\kappa}\tilde{d}$, $u = \delta^{\kappa-1}\tilde{u}$, and $\lambda = \delta\tilde{\lambda}$ in order to illuminate the fast-slow structure that the eigenvalue problem inherits from the wave itself. Recall that the traveling wave does not depend on the fast variable ξ, but only on the slow, $\eta = \delta\xi$, and super-slow, $\zeta = \delta^2\xi$, variables. System (4.52) becomes

$$\frac{d\tilde{U}}{d\xi} = \tilde{A}(\eta,\tilde{\lambda})\tilde{U}, \tag{4.55}$$

where $\tilde{U} = (p, \tilde{b}, q, \tilde{d}, \tilde{u})^T$,

$$\tilde{A}(\eta, \tilde{\lambda}) = \begin{pmatrix} -\delta\frac{2\tilde{v}(1-y)}{\tilde{K}_S} & \delta\frac{(1-y)^2}{\tilde{K}_S} & 0 & 0 & 0 \\ \frac{\tilde{\lambda}R_d\tilde{K}_S}{(1-y)^2} + \tilde{a}_1 wH & -(cR_d - 1) & \tilde{a}_1 yH & 0 & \tilde{a}_1 yw \\ 0 & 0 & -\delta\frac{2\tilde{r}(1-w)}{\tilde{K}_A} & \delta\frac{(1-w)^2}{\tilde{K}_A} & 0 \\ \tilde{a}_1 a_2 wH & 0 & \frac{\tilde{\lambda}\tilde{K}_A}{(1-w)^2} + \tilde{a}_1 a_2 yH & -(c-1) & \tilde{a}_1 a_2 yw \\ -\delta\frac{\tilde{a}_3}{c}wH & 0 & -\delta\frac{\tilde{a}_3}{c}yH & 0 & \delta(\frac{\tilde{\lambda}}{c} - \frac{\tilde{a}_3}{c}yw) + \delta^2\frac{\tilde{a}_4}{c} \end{pmatrix}, \tag{4.56}$$

$w = w_{tw}(\eta)$, $y = y_{tw}(\eta)$, $\tilde{v} = \tilde{v}_{tw}(\eta)$, $\tilde{r} = \tilde{r}_{tw}(\eta)$ and $H = H(\eta) \equiv \tilde{m}_{tw}(\eta) + \delta^{1-\kappa}$.

We would like to determine if there are any elements of $\sigma_{pt}(\mathcal{L})$ with $\mathrm{Re}\lambda > 0$. This is equivalent to determining if there are any bounded solutions to (4.55) for $\mathrm{Re}\lambda > 0$ (or, because the asymptotic matrices are hyperbolic for $\lambda \notin \sigma_{ess}$, if there are any solutions which decay to zero exponentially fast at $\pm\infty$). Our strategy in making this determination is as follows. We will exploit the fast-slow structure present in system (4.55) in order to determine for which values of λ a bounded solution exists.

Notice that in system (4.55) there are two fast variables, $\tilde{b}$ and $\tilde{d}$, and three slow variables, p, q, and $\tilde{u}$. In addition, the nonautonomous terms in the matrix $\tilde{A}$ depend only on the variables $\eta = \delta\xi$ and $\zeta = \delta\eta$, because the traveling wave only depends on these variables.

In other words, if we were to append equation (4.52) with the equation $\frac{d\eta}{d\xi} = \delta$, then the system would be autonomous, with two fast variables ($\tilde{b}$ and $\tilde{d}$) and four slow variables (p, q, $\tilde{u}$, and η). Hence, we may find an asymptotic expansion for the slow manifold

$$\begin{aligned} \tilde{b} &= G_0(p, q, \tilde{u}, \eta) + \delta^{1-\kappa}G_1(p, q, \tilde{u}, \eta) + \delta G_2(p, q, \tilde{u}, \eta) \\ \tilde{d} &= H_0(p, q, \tilde{u}, \eta) + \delta^{1-\kappa}H_1(p, q, \tilde{u}, \eta) + \delta H_2(p, q, \tilde{u}, \eta), \end{aligned} \tag{4.57}$$

where

$$\begin{aligned} G_0 &= \frac{1}{(cR_d - 1)}\left[\left(\frac{\tilde{\lambda}R_d\tilde{K}_S}{(1-y_{tw})^2} + \tilde{a}_1 w_{tw}\tilde{m}_{tw}\right)p + \tilde{a}_1 y_{tw}\tilde{m}_{tw}q + \tilde{a}_1 y_{tw}w_{tw}\tilde{u}\right] \\ H_0 &= \frac{1}{(c-1)}\left[\tilde{a}_1 a_2 w_{tw}\tilde{m}_{tw}p + \left(\frac{\tilde{\lambda}\tilde{K}_A}{(1-w_{tw})^2} + \tilde{a}_1 a_2 y_{tw}\tilde{m}_{tw}\right)q + \tilde{a}_1 a_2 y_{tw}\tilde{w}_{tw}\tilde{u}\right] \\ G_1 &= \frac{1}{(cR_d - 1)}\left(\tilde{a}_1 w_{tw}p + \tilde{a}_1 y_{tw}q\right) \\ H_1 &= \frac{1}{(c-1)}\left(\tilde{a}_1 a_2 w_{tw}p + \tilde{a}_1 a_2 y_{tw}q\right), \end{aligned} \tag{4.58}$$

and

$$
\begin{aligned}
-(cR_d-1)^2G_2 &= \left(\frac{\tilde{\lambda}R_d\tilde{K}_S}{(1-y_{tw})^2}+\tilde{a}_1w_{tw}\tilde{m}_{tw}\right)\left(\frac{-2\tilde{v}_{tw}(1-y_{tw})}{\tilde{K}_S}p+\frac{(1-y_{tw})^2}{\tilde{K}_S}G_0\right)\\
&\quad+\tilde{a}_1y_{tw}\tilde{m}_{tw}\left(\frac{-2\tilde{r}_{tw}(1-w_{tw})}{\tilde{K}_A}q+\frac{(1-w_{tw})^2}{\tilde{K}_A}H_0\right)\\
&\quad+\left(\frac{2\tilde{\lambda}R_d\tilde{K}_S\dot{y}_{tw}}{(1-y_{tw})^3}+\tilde{a}_1\dot{w}_{tw}\tilde{m}_{tw}+\tilde{a}_1w_{tw}\dot{\tilde{m}}_{tw}\right)p\\
&\quad+\left(\tilde{a}_1\dot{y}_{tw}\tilde{m}_{tw}+\tilde{a}_1y_{tw}\dot{\tilde{m}}_{tw}\right)q\\
-(1-c)^2H_2 &= \tilde{a}_1a_2w_{tw}\tilde{m}_{tw}\left(\frac{-2\tilde{v}_{tw}(1-y_{tw})}{\tilde{K}_S}p+\frac{(1-y_{tw})^2}{\tilde{K}_S}G_0\right)\\
&\quad+\left(\frac{\tilde{\lambda}\tilde{K}_A}{(1-w_{tw})^2}+\tilde{a}_1a_2y_{tw}\tilde{m}_{tw}\right)\left(\frac{-2\tilde{r}_{tw}(1-w_{tw})}{\tilde{K}_A}q+\frac{(1-w_{tw})^2}{\tilde{K}_A}H_0\right)\\
&\quad+\left(\tilde{a}_1a_2\dot{w}_{tw}\tilde{m}_{tw}+\tilde{a}_1a_2w_{tw}\dot{\tilde{m}}_{tw}\right)p\\
&\quad+\left(\frac{2\tilde{\lambda}\tilde{K}_A\dot{w}_{tw}}{(1-w_{tw})^3}+\tilde{a}_1a_2\dot{y}_{tw}\tilde{m}_{tw}+\tilde{a}_1a_2y_{tw}\dot{\tilde{m}}_{tw}\right)q.
\end{aligned}
\tag{4.59}
$$

In the above, $\dot{} = d/d\eta$.

The leading order fast dynamics off the slow manifold is given by

$$
\begin{aligned}
\tilde{b}_\xi &= -(cR_d-1)\tilde{b}+\left(\frac{\tilde{\lambda}R_d\tilde{K}_S}{(1-y_{tw})^2}+\tilde{a}_1w_{tw}\tilde{m}_{tw}\right)p+\tilde{a}_1y_{tw}\tilde{m}_{tw}q+\tilde{a}_1y_{tw}w_{tw}\tilde{u}\\
\tilde{d}_\xi &= -(c-1)\tilde{d}+\tilde{a}_1a_2w_{tw}\tilde{m}_{tw}p+\left(\frac{\tilde{\lambda}\tilde{K}_A}{(1-w_{tw})^2}+\tilde{a}_1a_2y_{tw}\tilde{m}_{tw}\right)q+\tilde{a}_1a_2y_{tw}w_{tw}\tilde{u},
\end{aligned}
\tag{4.60}
$$

where we recall that p, q, and $\tilde{u}$ are constant to leading order on $\mathcal{O}(1)$ intervals in the variable ξ. The leading order dynamics on the slow manifold are given by

$$\begin{aligned}
p_\eta &= \left(-\frac{2\tilde{v}_{tw}(1-y_{tw})}{\tilde{K}_S} + \frac{\tilde{\lambda}R_d}{(cR_d-1)} \right) p + \frac{\tilde{a}_1(1-y_{tw})^2}{\tilde{K}_S(cR_d-1)} \left(w_{tw}\tilde{m}_{tw}p + y_{tw}\tilde{m}_{tw}q + y_{tw}w_{tw}\tilde{u} \right) \\
&\quad + \delta^{1-\kappa}\frac{\tilde{a}_1(1-y_{tw})^2}{\tilde{K}_S(cR_d-1)} \left(w_{tw}p + y_{tw}q \right) + \delta\frac{(1-y_{tw})^2}{\tilde{K}_S}G_2 \\
q_\eta &= \left(-\frac{2\tilde{r}_{tw}(1-w_{tw})}{\tilde{K}_A} + \frac{\tilde{\lambda}}{(c-1)} \right) q + \frac{\tilde{a}_1 a_2(1-w_{tw})^2}{\tilde{K}_A(c-1)} \left(w_{tw}\tilde{m}_{tw}p + y_{tw}\tilde{m}_{tw}q + y_{tw}w_{tw}\tilde{u} \right) \\
&\quad + \delta^{1-\kappa}\frac{\tilde{a}_1 a_2(1-w_{tw})^2}{\tilde{K}_A(c-1)} \left(w_{tw}p + y_{tw}q \right) + \delta\frac{(1-w_{tw})^2}{\tilde{K}_A}H_2 \\
\tilde{u}_\eta &= \frac{\tilde{\lambda}}{c}\tilde{u} - \frac{\tilde{a}_3}{c}\left(w_{tw}\tilde{m}_{tw}p + y_{tw}\tilde{m}_{tw}q + y_{tw}w_{tw}\tilde{u} \right) - \delta^{1-\kappa}\frac{\tilde{a}_3}{c}\left(w_{tw}p + y_{tw}q \right) + \delta\frac{\tilde{a}_4}{c}\tilde{u}.
\end{aligned} \tag{4.61}$$

Analysis of the fast system

First, we consider equation (4.60). Because $(cR_d - 1) > 0$ and $(c-1) < 0$, there is one expanding and one contracting direction. To leading order, the slow variables, p, q, $\tilde{u}$, and η are constant. Thus, the invariant slow manifold is normally hyperbolic, and if a solution leaves this slow manifold, it must become unbounded in either forward or backward time. Thus, we may conclude that, in order to have a bounded eigenfunction, solutions must not leave the slow manifold. As a result, we need only study the behavior of solutions to system (4.61). This fact is somewhat expected, since the traveling wave itself is also restricted to a slow manifold (recall the discussion after equation (4.25)).

Construction of the Evans Function

We now define the Evans function for the three-dimensional system (4.61). We will denote this system by

$$\frac{d}{d\eta}\begin{pmatrix} p \\ q \\ \tilde{u} \end{pmatrix} = B(\eta;\tilde{\lambda})\begin{pmatrix} p \\ q \\ \tilde{u} \end{pmatrix}. \tag{4.62}$$

Recall that, up to and including terms of $\mathcal{O}(\delta)$, as $\eta \to \pm\infty$ all three components of the traveling wave have constant limits. They are given by

$$\begin{aligned}
y_{tw}(-\infty) &= 0 & y_{tw}(+\infty) &= \frac{1}{1+\delta^\kappa \tilde{K}_S} \\
w_{tw}(-\infty) &= \frac{1}{1+\delta^\kappa \tilde{K}_A} & w_{tw}(+\infty) &= 0 \\
\tilde{m}_{tw}(-\infty) &= 0 & \tilde{m}_{tw}(+\infty) &= 0.
\end{aligned} \tag{4.63}$$

We can use this information to compute the asymptotic matrices associated with system (4.61). At $\eta = +\infty$ we find

$$\lim_{\eta\to+\infty} B(\eta;\tilde{\lambda}) \equiv B^+(\tilde{\lambda}) =$$
$$\begin{pmatrix} \frac{\tilde{\lambda}R_d}{(cR_d-1)} - \delta\frac{\tilde{\lambda}^2R_d^2}{(cR_d-1)^3} & 0 & 0 \\ 0 & -\frac{\tilde{\lambda}}{a_2(cR_d-1)} - \delta^{1-\kappa}\frac{\tilde{a}_1}{\tilde{K}_A(cR_d-1)(1+\delta^\kappa\tilde{K}_S)} + \delta\frac{\tilde{\lambda}^2}{a_2^3(cR_d-1)^3} & 0 \\ 0 & -\delta^{1-\kappa}\frac{\tilde{a}_3}{c(1+\delta^\kappa\tilde{K}_S)} & \frac{\tilde{\lambda}}{c} \end{pmatrix}, \tag{4.64}$$

where we recall (4.11). The eigenvalues of this matrix are given, to $\mathcal{O}(\delta^{1-\kappa})$, by

$$\begin{aligned}
\nu_1^+(\tilde{\lambda}) &= \frac{\tilde{\lambda}R_d}{(cR_d-1)} \\
\nu_2^+(\tilde{\lambda}) &= -\frac{\tilde{\lambda}}{a_2(cR_d-1)} - \delta^{1-\kappa}\frac{\tilde{a}_1}{\tilde{K}_A(cR_d-1)(1+\delta^\kappa\tilde{K}_S)} \\
\nu_3^+(\tilde{\lambda}) &= \frac{\tilde{\lambda}}{c},
\end{aligned} \tag{4.65}$$

with eigenvectors

$$\begin{aligned}
e_1^+ &= (1,0,0)^t \\
e_2^+ &= (0,1,\delta^{1-\kappa}\frac{\tilde{a}_3}{c(1+\delta^\kappa\tilde{K}_S)(\frac{\tilde{\lambda}}{c}-\nu_2^+)})^t \\
e_3^+ &= (0,0,1)^t.
\end{aligned} \tag{4.66}$$

As a result, when $\mathrm{Re}(\tilde{\lambda}) > 0$ any solution that remains bounded as $\eta \to +\infty$ must be asymptotic to e_2^+. Because this is the unique stable direction, there exists a unique solution

$\gamma^+(\eta,\tilde{\lambda})$ of equation (4.61) satisfying

$$\lim_{\eta\to+\infty}\gamma^+(\eta,\tilde{\lambda})\exp(-\nu_2^+(\tilde{\lambda})\eta)=e_2^+. \tag{4.67}$$

Similarly, we compute

$$\lim_{\eta\to-\infty}B(\eta;\tilde{\lambda})\equiv B^-(\tilde{\lambda})=$$
$$\begin{pmatrix} \frac{\tilde{\lambda}R_d}{(cR_d-1)}+\delta^{1-\kappa}\frac{\tilde{a}_1}{\tilde{K}_S(cR_d-1)(1+\delta^\kappa\tilde{K}_A)}-\delta\frac{\tilde{\lambda}^2R_d^2}{(cR_d-1)^3} & 0 & 0\\ 0 & -\frac{\tilde{\lambda}}{a_2(cR_d-1)}+\delta\frac{\tilde{\lambda}^2}{a_2^3(cR_d-1)^3} & 0\\ -\delta^{1-\kappa}\frac{\tilde{a}_3}{c(1+\delta^\kappa\tilde{K}_A)} & 0 & \frac{\tilde{\lambda}}{c}\end{pmatrix}. \tag{4.68}$$

The eigenvalues of this matrix are, to leading order

$$\begin{aligned}\nu_1^- &= \frac{\tilde{\lambda}R_d}{(cR_d-1)}+\delta^{1-\kappa}\frac{\tilde{a}_1}{\tilde{K}_S(cR_d-1)(1+\delta^\kappa\tilde{K}_A)}\\ \nu_2^- &= -\frac{\tilde{\lambda}}{a_2(cR_d-1)}\\ \nu_3^- &= \frac{\tilde{\lambda}}{c},\end{aligned} \tag{4.69}$$

with eigenvectors

$$\begin{aligned}e_1^- &= (1,0,\delta^{1-\kappa}\frac{\tilde{a}_3}{c(1+\delta^\kappa\tilde{K}_A)(\frac{\tilde{\lambda}}{c}-\nu_1^-)})^t\\ e_2^- &= (0,1,0)^t\\ e_3^- &= (0,0,1)^t.\end{aligned} \tag{4.70}$$

Therefore, for $\mathrm{Re}(\tilde{\lambda})>0$, any solution that remains bounded as $\eta\to-\infty$ must be asymptotic to e_1^- or e_3^-. It is possible to find two solutions to (4.61), $\alpha^-(\eta,\tilde{\lambda})$ and $\beta^-(\eta,\tilde{\lambda})$, such that

$$\begin{aligned}\lim_{\eta\to-\infty}\alpha^-(\eta)\exp(-\nu_1^-(\tilde{\lambda})\eta) &= e_1^-\\ \lim_{\eta\to-\infty}\beta^-(\eta)\exp(-\nu_3^-(\tilde{\lambda})\eta) &= e_3^-.\end{aligned} \tag{4.71}$$

Define the Evans function [1] as

$$D(\tilde{\lambda}) = e^{-\int_0^\eta \mathrm{Tr}B(s;\tilde{\lambda})ds} \det[\alpha^-(\eta,\tilde{\lambda}), \beta^-(\eta,\tilde{\lambda}), \gamma^+(\eta,\tilde{\lambda})], \tag{4.72}$$

where we note that, as explained in section 4.1, D is independent of η. Zeros of the Evans function correspond to eigenvalues of equation (4.61), and so in order to prove that the traveling wave is (spectrally) stable we must simply show that $D(\tilde{\lambda})$ has no zeros in the right half of the complex plane.

It turns out that the exponential factor in the definition of $D(\lambda)$ is explicitly computable. Up to and including $\mathcal{O}(1, \delta^{1-\kappa})$, we have

$$\begin{aligned}
\mathrm{Tr}B(\eta;\tilde{\lambda}) = & -\frac{2\tilde{v}_{tw}(1-y_{tw})}{\tilde{K}_S} + \frac{\tilde{\lambda}R_d}{(cR_d-1)} + \frac{\tilde{a}_1(1-y_{tw})^2}{\tilde{K}_S(cR_d-1)} w_{tw}(\tilde{m}_{tw}+\delta^{1-\kappa}) \\
& - \frac{2\tilde{r}_{tw}(1-w_{tw})}{\tilde{K}_A} + \frac{\tilde{\lambda}}{(c-1)} + \frac{\tilde{a}_1 a_2(1-w_{tw})^2}{\tilde{K}_A(c-1)} y_{tw}(\tilde{m}_{tw}+\delta^{1-\kappa}) \\
& + \frac{\tilde{\lambda}}{c} - \frac{\tilde{a}_3}{c} y_{tw} w_{tw} \\
= & -\frac{2y_\eta}{1-y} + \frac{\tilde{\lambda}R_d}{(cR_d-1)} + \frac{y_\eta}{y} - \frac{2w_\eta}{1-w} + \frac{\tilde{\lambda}}{(c-1)} + \frac{w_\eta}{w} + \frac{\tilde{\lambda}}{c} + \frac{\tilde{m}_\eta}{\tilde{m}-\delta^{1-\kappa}},
\end{aligned}$$

where we used (4.27). Therefore, we find that

$$\begin{aligned}
D(\tilde{\lambda}) = & \frac{y(0)(1-y(0))^2 w(0)(1-w(0))^2(\tilde{m}(0)+\delta^{1-\kappa})}{(1-y(\eta))^2(1-w(\eta))^2(\tilde{m}(\eta)+\delta^{1-\kappa})} \\
& \times \det[\frac{e^{-\frac{\tilde{\lambda}R_d}{cR_d-1}\eta}}{y(\eta)}\alpha^-(\eta,\tilde{\lambda}), e^{-\frac{\tilde{\lambda}}{c}\eta}\beta^-(\eta,\tilde{\lambda}), \frac{e^{\frac{\tilde{\lambda}}{a_2(cR_d-1)}\eta}}{w(\eta)}\gamma^+(\eta,\tilde{\lambda})].
\end{aligned} \tag{4.73}$$

In the above expression we have chosen to multiply α^-, β^-, and γ^+ by certain terms that result from the exponential of the trace of the matrix B. The reason for this is as follows. Because $D(\tilde{\lambda})$ is independent of η, we may evaluate the right hand side of equation (4.73) at any value of η that we like. In particular, we can let $\eta \to -\infty$. The reason this is useful is that

$$\lim_{\eta\to-\infty} e^{-\nu_2^-(\tilde{\lambda})\eta}\gamma^+(\eta,\tilde{\lambda}) = t(\tilde{\lambda})e_2^-, \tag{4.74}$$

where $t(\tilde{\lambda})$ is known as the transmission function. If, and only if, $t(\tilde{\lambda})$ is zero, then $\gamma^+ \to 0$ as $\eta \to -\infty$. Hence, $\tilde{\lambda}$ is an eigenvalue if and only if $t(\tilde{\lambda}) = 0$. Furthermore, using equation

(4.27), we see that

$$y(\eta) \sim e^{\delta^{1-\kappa} \frac{\tilde{a}_1}{\tilde{K}_S(cR_d-1)(1+\delta^\kappa \tilde{K}_A)}\eta}, \text{ as } \eta \to -\infty.$$

Thus,

$$\frac{e^{-\frac{\tilde{\lambda} R_d}{cR_d-1}\eta}}{y(\eta)} \sim e^{-\nu_1^-(\tilde{\lambda})\eta} \text{ as } \eta \to -\infty,$$

and, if we take the limit as $\eta \to -\infty$ in equation (4.73), we see that

$$\begin{aligned} D(\tilde{\lambda}) &= \frac{y(0)(1-y(0))^2 w(0)(1-w(0))^2(\tilde{m}(0)+\delta^{1-\kappa})}{(1-y(-\infty))^2(1-w(-\infty))^2(\tilde{m}(-\infty)+\delta^{1-\kappa})} \\ &\quad \times \det[e_1^-, e_3^-, \frac{t(\tilde{\lambda})}{w(-\infty)} e_2^-] \\ &= \left[\delta^{2\kappa} \frac{(1+\delta^\kappa \tilde{K}_A)^2(\tilde{K}_A+\tilde{K}_S)^3}{\tilde{K}_A(1+\delta^\kappa \tilde{K}_S+\delta^\kappa \tilde{K}_A)^5} + \mathcal{O}(\delta^{1+2\kappa})\right] t(\tilde{\lambda}). \end{aligned} \tag{4.75}$$

Hence, if we determine the zeros of $t(\tilde{\lambda})$ we determine the zeros of $D(\tilde{\lambda})$.

Locating the zeros of the transmission function

It turns out that, using the structure of equation (4.61), we will be able to determine an explicit expression for $t(\tilde{\lambda})$. Up to and including terms of $\mathcal{O}(1, \delta^{1-\kappa})$, system (4.61) satisfies:

$$\begin{aligned} &\frac{\tilde{K}_S}{(1-y)^2}\left[p_\eta + \left(\frac{2\tilde{v}(1-y)}{\tilde{K}_S} - \frac{\tilde{\lambda} R_d}{(cR_d-1)}\right) p\right] \\ &\qquad = -\frac{\tilde{K}_A}{(1-w)^2}\left[q_\eta + \left(\frac{2\tilde{r}(1-w)}{\tilde{K}_A} + \frac{\tilde{\lambda}}{a_2(cR_d-1)}\right) q\right] \\ &\qquad = -\frac{\tilde{a}_1 c}{\tilde{a}_3(cR_d-1)}\left[\tilde{u}_\eta - \frac{\tilde{\lambda}}{c}\tilde{u}\right]. \end{aligned}$$

By simplifying this relationship, we obtain

$$\begin{aligned}
\tilde{K}_S e^{\frac{\tilde{\lambda} R_d}{(cR_d-1)}\eta} \frac{d}{d\eta}\left[e^{-\frac{\tilde{\lambda} R_d}{(cR_d-1)}\eta} \frac{p(\eta)}{(1-y(\eta))^2}\right] \\
= -\tilde{K}_A e^{-\frac{\tilde{\lambda}}{a_2(cR_d-1)}\eta} \frac{d}{d\eta}\left[e^{\frac{\tilde{\lambda}}{a_2(cR_d-1)}\eta} \frac{q(\eta)}{(1-w(\eta))^2}\right] \\
= -\frac{\tilde{a}_1 c}{\tilde{a}_3(cR_d-1)} e^{\frac{\tilde{\lambda}}{c}\eta} \frac{d}{d\eta}\left[e^{-\frac{\tilde{\lambda}}{c}\eta} \tilde{u}(\eta)\right].
\end{aligned} \tag{4.76}$$

This is a relationship that any solution to (4.61) must satisfy, up to and including terms of $\mathcal{O}(1, \delta^{1-\kappa})$. If we multiply the relation (4.76) by $e^{\frac{\tilde{\lambda}}{a_2(cR_d-1)}\eta}$, integrate from $\eta_0 \to \eta$, and use integration by parts, we obtain

$$\begin{aligned}
-\tilde{K}_A \left(e^{\frac{\tilde{\lambda}}{a_2(cR_d-1)}\eta} \frac{q(\eta)}{(1-w(\eta))^2} - e^{\frac{\tilde{\lambda}}{a_2(cR_d-1)}\eta_0} \frac{q(\eta_0)}{(1-w(\eta_0))^2}\right) \\
= -\frac{\tilde{a}_1 c}{\tilde{a}_3(cR_d-1)} \left(e^{\frac{\tilde{\lambda}}{a_2(cR_d-1)}\eta} \tilde{u}(\eta) - e^{\frac{\tilde{\lambda}}{a_2(cR_d-1)}\eta_0} \tilde{u}(\eta_0)\right) \\
+ \frac{\tilde{a}_1 c}{\tilde{a}_3(cR_d-1)} \left(\frac{\tilde{\lambda}}{a_2(cR_d-1)} + \frac{\tilde{\lambda}}{c}\right) \int_{\eta_0}^{\eta} e^{\frac{\tilde{\lambda}}{a_2(cR_d-1)}s} \tilde{u}(s) ds \\
= \tilde{K}_S \left(e^{\frac{\tilde{\lambda}}{a_2(cR_d-1)}\eta} \frac{p(\eta)}{(1-y(\eta))^2} - e^{\frac{\tilde{\lambda}}{a_2(cR_d-1)}\eta_0} \frac{p(\eta_0)}{(1-y(\eta_0))^2}\right) \\
- \tilde{K}_S \left(\frac{\tilde{\lambda}}{a_2(cR_d-1)} + \frac{\tilde{\lambda} R_d}{(cR_d-1)}\right) \int_{\eta_0}^{\eta} e^{\frac{\tilde{\lambda}}{a_2(cR_d-1)}s} \frac{p(s)}{(1-y(s))^2} ds.
\end{aligned} \tag{4.77}$$

If we now consider the solution $\gamma^+(\eta, \tilde{\lambda})$, we know that

$$\begin{aligned}
\lim_{\eta \to +\infty} \gamma^+(\eta, \tilde{\lambda}) \exp(-\nu_2^+(\tilde{\lambda})\eta) = e_2^+ \\
\lim_{\eta \to -\infty} \gamma^+(\eta, \tilde{\lambda}) \exp(-\nu_2^-(\tilde{\lambda})\eta) = t(\tilde{\lambda}) e_2^-,
\end{aligned} \tag{4.78}$$

where the function $t(\tilde{\lambda})$ is the transmission function defined in equation (4.74). By sending $\eta \to +\infty$ and $\eta_0 \to -\infty$ in equation (4.77), we find that

$$\begin{aligned}
t(\tilde{\lambda}) = \tilde{\lambda} \left(\frac{\delta^{2\kappa} \tilde{K}_A}{(1+\delta^\kappa \tilde{K}_A)^2}\right) \left(\frac{\tilde{a}_1 (a_2 R_d - 1)^2}{\tilde{a}_3 a_2 (R_d-1)^2}\right) \int_{-\infty}^{+\infty} e^{\frac{\tilde{\lambda}}{a_2(cR_d-1)}s} \tilde{u}_\gamma(s) ds \\
= -\tilde{\lambda} \left(\frac{\delta^{2\kappa} \tilde{K}_A}{(1+\delta^\kappa \tilde{K}_A)^2}\right) \left(\frac{\tilde{K}_S (a_2 R_d - 1)^2}{a_2(R_d-1)}\right) \int_{-\infty}^{+\infty} e^{\frac{\tilde{\lambda}}{a_2(cR_d-1)}s} \frac{p_\gamma(s)}{(1-y(s))^2} ds,
\end{aligned} \tag{4.79}$$

where $\tilde{u}_\gamma$ and p_γ denote the $\tilde{u}$ and p components of the solution γ^+. Here we have used the fact that

$$\begin{aligned}
&\lim_{\eta\to\infty} \gamma^+(\eta) e^{\frac{\tilde{\lambda}}{a_2(cR_d-1)}\eta} \\
&= \lim_{\eta\to\infty} \gamma^+(\eta) e^{\left(\frac{\tilde{\lambda}}{a_2(cR_d-1)} + \delta^{1-\kappa}\frac{\tilde{a}_1}{\tilde{K}_A(cR_d-1)(1+\delta^\kappa \tilde{K}_S)}\right)\eta} e^{-\delta^{1-\kappa}\frac{\tilde{a}_1}{\tilde{K}_A(cR_d-1)(1+\delta^\kappa\tilde{K}_S)}\eta} \\
&= e_2^+ \cdot 0 = 0.
\end{aligned}$$

In addition, we may use the relationship (4.76) to write both p and q in terms of $\tilde{u}$. In particular, for $q(\eta)$, if we multiply (4.76) by $e^{\frac{\tilde{\lambda}}{a_2(cR_d-1)}\eta}$ and integrate from η to η_0, we find that

$$\begin{aligned}
-\tilde{K}_A &\left(e^{\frac{\tilde{\lambda}}{a_2(cR_d-1)}\eta_0} \frac{q(\eta_0)}{(1-w(\eta_0))^2} - e^{\frac{\tilde{\lambda}}{a_2(cR_d-1)}\eta}\frac{q(\eta)}{(1-w(\eta))^2}\right) \\
&= -\frac{\tilde{a}_1 c}{\tilde{a}_3(cR_d-1)}\left(e^{\frac{\tilde{\lambda}}{a_2(cR_d-1)}\eta_0}\tilde{u}(\eta_0) - e^{\frac{\tilde{\lambda}}{a_2(cR_d-1)}\eta}\tilde{u}(\eta)\right) \\
&\quad + \frac{\tilde{a}_1 c}{\tilde{a}_3(cR_d-1)}\left(\frac{\tilde{\lambda}}{a_2(cR_d-1)} + \frac{\tilde{\lambda}}{c}\right)\int_\eta^{\eta_0} e^{\frac{\tilde{\lambda}}{a_2(cR_d-1)}s}\tilde{u}(s)ds.
\end{aligned}$$

Taking the limit as $\eta_0 \to \infty$ and using (4.78), we find that

$$\begin{aligned}
q_\gamma(\eta) = &\frac{\tilde{a}_1 c}{\tilde{K}_A \tilde{a}_3(cR_d-1)}(1-w(\eta))^2\tilde{u}_\gamma(\eta) \\
&+ \frac{\tilde{a}_1 c}{\tilde{K}_A\tilde{a}_3(cR_d-1)}\left(\tilde{\lambda}\frac{[c+a_2(cR_d-1)]}{ca_2(cR_d-1)}\right)\times \\
&\times (1-w(\eta))^2 e^{-\frac{\tilde{\lambda}}{a_2(cR_d-1)}\eta}\int_\eta^\infty e^{\frac{\tilde{\lambda}}{a_2(cR_d-1)}s}\tilde{u}_\gamma(s)ds.
\end{aligned} \tag{4.80}$$

Similarly, we can multiply (4.76) by $e^{-\frac{\tilde{\lambda}R_d}{(cR_d-1)}\eta}$, integrate from η to η_0, and let $\eta_0 \to \infty$ to find that

$$\begin{aligned}
p_\gamma(\eta) = &-\frac{\tilde{a}_1 c}{\tilde{K}_S\tilde{a}_3(cR_d-1)}(1-y(\eta))^2\tilde{u}_\gamma(\eta) \\
&+ \frac{\tilde{a}_1 c}{\tilde{K}_S\tilde{a}_3(cR_d-1)}\left(\frac{\tilde{\lambda}R_d}{(cR_d-1)} - \frac{\tilde{\lambda}}{c}\right)(1-y(\eta))^2 e^{\frac{\tilde{\lambda}R_d}{(cR_d-1)}\eta}\int_\eta^\infty e^{-\frac{\tilde{\lambda}R_d}{(cR_d-1)}s}\tilde{u}_\gamma(s)ds.
\end{aligned} \tag{4.81}$$

If we now insert expressions (4.80) and (4.81) into equation (4.61), we find that the reduced eigenvalue problem is given by

$$
\begin{aligned}
t(\tilde{\lambda}) &= \tilde{\lambda}\left(\frac{\delta^{2\kappa}\tilde{K}_A}{(1+\delta^{\kappa}\tilde{K}_A)^2}\right)\left(\frac{\tilde{a}_1(a_2R_d-1)^2}{\tilde{a}_3a_2}\right)\int_{-\infty}^{+\infty} e^{\frac{\tilde{\lambda}}{a_2(cR_d-1)}s}\tilde{u}_\gamma(s)ds \\
\frac{d}{d\eta}\tilde{u}_\gamma &= \left(\frac{\tilde{\lambda}}{c}+\frac{y_\eta}{y}+\frac{w_\eta}{w}+\frac{\tilde{m}_\eta}{\tilde{m}+\delta^{1-\kappa}}\right)\tilde{u}_\gamma \\
&\quad -\frac{\tilde{\lambda}}{c(cR_d-1)}\frac{y_\eta}{y}e^{\frac{\tilde{\lambda}R_d}{(cR_d-1)}\eta}\int_\eta^\infty e^{-\frac{\tilde{\lambda}R_d}{(cR_d-1)}s}\tilde{u}_\gamma(s)ds \\
&\quad +\frac{\tilde{\lambda}(c+a_2(cR_d-1))}{ca_2(cR_d-1)}\frac{w_\eta}{w}e^{-\frac{\tilde{\lambda}}{a_2(cR_d-1)}\eta}\int_\eta^\infty e^{\frac{\tilde{\lambda}}{a_2(cR_d-1)}s}\tilde{u}_\gamma(s)ds.
\end{aligned} \tag{4.82}
$$

Here we have used the equations for y_η, w_η, and $\tilde{m}_\eta$ given in equation (4.27) to simplify the above expression. Note that one could also define the reduced eigenvalue problem in terms of the p-component of the eigenfunction.

We remark that, using equation (4.82), one can see explicitly that $t(0) = 0$. This is due to the fact that $\tilde{\lambda} = 0$ is a eigenvalue with eigenfunction given by the derivative of the traveling wave. Furthermore, one can see that, for $\tilde{\lambda} = 0$, the solution $\tilde{u}(\eta)$ to equation (4.82) is exactly $\frac{d}{d\eta}\tilde{m}_{tw}$, where $\frac{d}{d\eta}\tilde{m}_{tw}$ is given in equation (4.27).

We now indicate how equation (4.82) can be used to show that if $\mathrm{Re}(\tilde{\lambda}) > 0$ then $t(\tilde{\lambda}) \neq 0$, which implies that the traveling wave is spectrally stable. Define a new variable z as

$$
z(\eta) = \frac{e^{-\frac{\tilde{\lambda}}{c}\eta}\tilde{u}_\gamma(\eta)}{y(\eta)w(\eta)(\tilde{m}(\eta)+\delta^{1-\kappa})}. \tag{4.83}
$$

We then see that

$$
\begin{aligned}
z_\eta = &-\frac{\tilde{\lambda}}{c(cR_d-1)}\frac{y_\eta e^{\left(-\frac{\tilde{\lambda}}{c}+\frac{\tilde{\lambda}R_d}{(cR_d-1)}\right)\eta}}{y^2w(\tilde{m}+\delta^{1-\kappa})}\int_\eta^\infty e^{\left(\frac{\tilde{\lambda}}{c}-\frac{\tilde{\lambda}R_d}{(cR_d-1)}\right)s}yw(\tilde{m}+\delta^{1-\kappa})z(s)ds \\
&+\frac{\tilde{\lambda}(c+a_2(cR_d-1))}{ca_2(cR_d-1)}\frac{w_\eta e^{-\left(\frac{\tilde{\lambda}}{c}+\frac{\tilde{\lambda}}{a_2(cR_d-1)}\right)\eta}}{yw^2(\tilde{m}+\delta^{1-\kappa})}\int_\eta^\infty e^{\left(\frac{\tilde{\lambda}}{c}+\frac{\tilde{\lambda}}{a_2(cR_d-1)}\right)s}yw(\tilde{m}+\delta^{1-\kappa})z(s)ds.
\end{aligned} \tag{4.84}
$$

We now argue that on each half line, $\eta \geq 0$ and $\eta \leq 0$, one of the nonlocal terms can be neglected, to leading order, and we use this fact to show that there can be no eigenvalues with positive real part. First consider $\eta \geq 0$. Here $y = 1-\mathcal{O}(\delta^\kappa)$, and so $(1-y)^2 = \mathcal{O}(\delta^{2\kappa})$. Therefore, to leading order the nonlocal term with a factor of y_η in front of it is small.

Retaining the other nonlocal term, equation (4.84) for $\eta \geq 0$ may be written

$$\begin{aligned} &\frac{d}{d\eta}\left[\frac{w}{(1-w)^2} z_\eta e^{\left(\frac{\tilde{\lambda}}{c}+\frac{\tilde{\lambda}}{a_2(cR_d-1)}\right)\eta}\right] \\ &\quad = \frac{\tilde{a}_1}{\tilde{K}_A(cR_d-1)} \frac{\tilde{\lambda}(c+a_2(cR_d-1))}{ca_2(cR_d-1)} yw(\tilde{m}+\delta^{1-\kappa}) z e^{\left(\frac{\tilde{\lambda}}{c}+\frac{\tilde{\lambda}}{a_2(cR_d-1)}\right)\eta}. \end{aligned} \tag{4.85}$$

By multiplying this equation by z and integrating from 0 to $+\infty$, we obtain

$$\begin{aligned} \frac{w}{(1-w)^2} z z_\eta e^{\left(\frac{\tilde{\lambda}}{c}+\frac{\tilde{\lambda}}{a_2(cR_d-1)}\right)\eta}\Big|_0^\infty &= \int_0^\infty \frac{w}{(1-w)^2} z_\eta^2(\eta) e^{\left(\frac{\tilde{\lambda}}{c}+\frac{\tilde{\lambda}}{a_2(cR_d-1)}\right)\eta} d\eta \\ &+ \frac{\tilde{a}_1}{\tilde{K}_A(cR_d-1)} \frac{\tilde{\lambda}(c+a_2(cR_d-1))}{ca_2(cR_d-1)} \int_0^\infty yw(\tilde{m}+\delta^{1-\kappa}) z^2(\eta) e^{\left(\frac{\tilde{\lambda}}{c}+\frac{\tilde{\lambda}}{a_2(cR_d-1)}\right)\eta} d\eta. \end{aligned}$$

One can check that the boundary term at $\eta = +\infty$ must be zero (if z represents an eigenfunction). Thus, we have that

$$\begin{aligned} \frac{w(0)}{(1-w(0))^2} z(0) z_\eta(0) &= -\int_0^\infty \frac{w}{(1-w)^2} z_\eta^2(\eta) e^{\left(\frac{\tilde{\lambda}}{c}+\frac{\tilde{\lambda}}{a_2(cR_d-1)}\right)\eta} d\eta \\ &- \frac{\tilde{a}_1}{\tilde{K}_A(cR_d-1)} \frac{\tilde{\lambda}(c+a_2(cR_d-1))}{ca_2(cR_d-1)} \int_0^\infty yw(\tilde{m}+\delta^{1-\kappa}) z^2(\eta) e^{\left(\frac{\tilde{\lambda}}{c}+\frac{\tilde{\lambda}}{a_2(cR_d-1)}\right)\eta} d\eta. \end{aligned} \tag{4.86}$$

Hence, unless $z \equiv 0$, $z(0)z_\eta(0) < 0$.

Similarly, one can argue that the nonlocal term with a factor of w_η in front of it is small when $\eta \leq 0$, because $(1-w)^2 = \mathcal{O}(\delta^{2\kappa})$ there. A similar calculation as above leads to

$$\begin{aligned} \frac{y}{(1-y)^2} z z_\eta e^{\left(\frac{\tilde{\lambda}}{c}-\frac{\tilde{\lambda}R_d}{(cR_d-1)}\right)\eta}\Big|_{-\infty}^0 &= +\int_{-\infty}^0 \frac{y}{(1-y)^2} z_\eta^2(\eta) e^{\left(\frac{\tilde{\lambda}}{c}-\frac{\tilde{\lambda}R_d}{(cR_d-1)}\right)\eta} d\eta \\ &+ \frac{\tilde{a}_1}{\tilde{K}_S(cR_d-1)} \frac{\tilde{\lambda}}{c(cR_d-1)} \int_{-\infty}^0 yw(\tilde{m}+\delta^{1-\kappa}) z^2(\eta) e^{\left(\frac{\tilde{\lambda}}{c}-\frac{\tilde{\lambda}R_d}{(cR_d-1)}\right)\eta} d\eta. \end{aligned}$$

In this case, however, one must be a bit more careful in dealing with the boundary term. One can explicitly check that, if z corresponds to an eigenfunction that approaches the strong unstable direction as $\eta \to -\infty$, then the boundary term at $-\infty$ will be zero. If z corresponds to an eigenfunction that approaches the weak unstable direction, however, then this boundary term is ill defined.

Suppose for the moment that the eigenfunction approaches the strong unstable direction. Then the above equation implies

$$
\begin{aligned}
\frac{y(0)}{(1-y(0))^2} z(0) z_\eta(0) = &+ \int_{-\infty}^{0} \frac{y}{(1-y)^2} z_\eta^2(\eta) e^{\left(\frac{\tilde{\lambda}}{c} - \frac{\tilde{\lambda} R_d}{(cR_d - 1)}\right)\eta} d\eta \\
&+ \frac{\tilde{a}_1}{\tilde{K}_S(cR_d-1)} \frac{\tilde{\lambda}}{c(cR_d-1)} \int_{-\infty}^{0} yw(\tilde{m} + \delta^{1-\kappa}) z^2(\eta) e^{\left(\frac{\tilde{\lambda}}{c} - \frac{\tilde{\lambda} R_d}{(cR_d - 1)}\right)\eta} d\eta,
\end{aligned}
\tag{4.87}
$$

and $z(0)z_\eta(0) > 0$ unless $z \equiv 0$, which is a contradiction. Therefore, there can be no eigenfunctions that connect the stable manifold at $\eta = +\infty$ to the strong unstable manifold at $-\infty$. Although more work is required to rule out any unstable eigenvalues that correspond to eigenfunctions connecting the stable manifold at $\eta = +\infty$ with the weak unstable manifold at $-\infty$, we expect that this can be shown similarly, by utilizing the relationship (4.76) and the form of the traveling wave itself.

Super-slow dynamics

Finally, we remark on the role of the super-slow dynamics in the stability analysis. Recall that, in the existence construction, it was shown that the y- and w-components of the traveling wave are independent of the super-slow variable ζ, while the $\tilde{m}$-component depends on ζ only as $\zeta \to -\infty$, *i.e.* near $\eta = -\infty$. As a result, we expect the p- and q- components of an eigenfunction to be independent of the super slow variable, while the $\tilde{u}$-component may depend on ζ near $\eta = -\infty$.

In the above stability argument, the analysis focused mainly on the $\tilde{u}$-component of the eigenfunction, and did not involve the super slow dynamics. The leading order, $\mathcal{O}(1, \delta^{1-\kappa})$, analysis implied that any eigenfunction corresponding to an unstable eigenvalue must be identically zero, independent of any super-slow behavior. Thus, the super-slow dynamics do not play a significant role in the analysis.

Because we know that when $\tilde{\lambda} = 0$ the corresponding eigenfunction is $(p, q, \tilde{u}) = \partial_\eta(y, w, \tilde{m})$, we include the specific analysis of this solution for completeness. When $\tilde{\lambda} = 0$, the relationship (4.76) simplifies, and we can explicitly determine the leading order integral curves for the eigenfunction:

$$
\begin{aligned}
\frac{\tilde{K}_S p(\eta)}{(1-y(\eta))^2} + \frac{\tilde{K}_A q(\eta)}{(1-w(\eta))^2} &= C_0 \\
\frac{\tilde{K}_S p(\eta)}{(1-y(\eta))^2} + \frac{\tilde{a}_1 c}{\tilde{a}_3(cR_d-1)} \tilde{u}(\eta) &= C_1 \\
\frac{\tilde{K}_A q(\eta)}{(1-w(\eta))^2} - \frac{\tilde{a}_1 c}{\tilde{a}_3(cR_d-1)} \tilde{u}(\eta) &= C_2.
\end{aligned}
\tag{4.88}
$$

In the above equation, $C_0 = C_1 + C_2$. Furthermore, because $\partial_\eta(y, w, \tilde{m}) \to 0$ as $\eta \to +\infty$, we know that $C_0 = C_1 = C_2 = 0$.

For $\eta > \frac{1}{\sqrt{\delta}}$, ie $\zeta > \sqrt{\delta}$, the super-slow dynamics are given by

$$\begin{aligned}
\delta p_\zeta &= \delta^{1+\kappa}\frac{\tilde{a}_1\tilde{K}_S}{(cR_d-1)(1+\delta^\kappa\tilde{K}_S)^3}q + \mathcal{O}(\delta^3) \\
\delta q_\zeta &= -\delta^{1-\kappa}\frac{\tilde{a}_1}{(cR_d-1)\tilde{K}_A(1+\delta^\kappa\tilde{K}_S)}q + \mathcal{O}(\delta^3) \\
\delta u_\zeta &= -\delta^{1-\kappa}\frac{\tilde{a}_3}{c(1+\delta^\kappa\tilde{K}_S)}q + \delta\frac{\tilde{a}_4}{c}\tilde{u}.
\end{aligned} \tag{4.89}$$

Balancing terms implies that $q \equiv 0$ here, and the super-slow dynamics are given by

$$\begin{aligned}
p_\zeta &= \mathcal{O}(\delta^2) \\
\tilde{u}_\zeta &= \frac{\tilde{a}_4}{c}\tilde{u}.
\end{aligned}$$

Hence, $p = \tilde{u} \equiv 0$ here.

When $\eta < -\frac{1}{\sqrt{\delta}}$, ie $\zeta < -\sqrt{\delta}$, the super-slow dynamics are given by

$$\begin{aligned}
\delta p_\zeta &= \frac{\tilde{a}_1(\tilde{m}(\zeta)+\delta^{1-\kappa})}{\tilde{K}_S(cR_d-1)(1+\delta^\kappa\tilde{K}_A)}p - \mathcal{O}(\delta^{1-2\kappa})p \\
\delta q_\zeta &= -\delta^{2\kappa}\frac{\tilde{a}_1\tilde{K}_A(\tilde{m}(\zeta)+\delta^{1-\kappa})}{(cR_d-1)(1+\delta^\kappa\tilde{K}_A)^3}p + \mathcal{O}(\delta)p \\
\delta u_\zeta &= -\frac{\tilde{a}_3}{c(1+\delta^\kappa\tilde{K}_A)}(\tilde{m}(\zeta)+\delta^{1-\kappa})p + \delta\frac{\tilde{a}_4}{c}\tilde{u}.
\end{aligned} \tag{4.90}$$

A balancing argument implies that $p \equiv 0$ here, and that the super-slow dynamics are given by

$$\begin{aligned}
q_\zeta &= \mathcal{O}(\delta^2) \\
\tilde{u}_\zeta &= \frac{\tilde{a}_4}{c}\tilde{u}.
\end{aligned}$$

This, in turn, implies that $q \equiv 0$ here, but $\tilde{u}$ could decay exponentially as $\zeta \to -\infty$. This behavior is exactly that of the derivative of the wave.

We remark that, although there is nontrivial super-slow behavior in the $\tilde{u}$ component of the eigenfunction, it does not match in a continuous manner with the leading order slow behavior of $\tilde{u}$. This is because $\lim_{\eta\to-\infty}\tilde{u}(\eta) = 0$, while $\lim_{\zeta\to 0}\tilde{u}(\zeta) \neq 0$. In order to complete this matching, we would need to include even higher order ($\ll \mathcal{O}(\delta)$) terms in the analysis. Nevertheless, the leading order, slow components of the eigenfunction decay to zero at $\eta = \pm\infty$, indicating that $\tilde{\lambda} = 0$ is indeed an eigenvalue of the linear operator.

Chapter 5

Nonlinear Partial Differential Equations: preliminary notions

In this chapter we briefly explain results regarding the existence and uniqueness of solutions to nonlinear PDEs and the existence of invariant manifolds.

5.1 Existence and Uniqueness of Solutions

We saw in chapter 2 that, if the operator associated to a linear PDE is the generator of a strongly continuous semigroup, then we may use that semigroup to represent solutions to the PDE. In this chapter, we will see how the semigroup may be used to represent solutions to nonlinear PDEs, as well.

Because the semigroup representation of solutions to nonlinear PDEs is similar to that of ODEs, we first recall the corresponding result in finite dimensions. Consider the ODE

$$u_t = Au + N(u), \quad u = u(t) \in \mathbb{R}^n, \quad u(0) = u_0.$$

The solution to this equation may be represented using the variation of constants formulation of the solution,

$$u(t) = e^{tA}u_0 + \int_0^t e^{(t-s)A}N(u(s))ds.$$

It can be proven that, under certain assumptions on the nonlinearity $N(u)$, there exists a $T > 0$ such that the above solution exists for all $t \in [0, T)$. Furthermore, if $T < \infty$, then $|u(t)| \to \infty$ as $t \to T$.

It turns out that, if the linear operator is the generator of a strongly continuous semigroup, we can represent solutions to nonlinear PDEs in an analogous manner. Consider now the nonlinear PDE

$$u_t = Au + N(u), \quad u = u(t) \in X, \quad u(0) = u_0, \tag{5.1}$$

where X is some infinite dimensional Banach space. One key difference between the nonlinear PDE and nonlinear ODE is that, for PDEs, one must be precise about the meaning of the term "solution." In fact, there are several types of solutions, two of which are defined as follows.

Definition 5.1.1 *A* **classical solution** *to equation (5.1) is a function* $u : [0, T] \to X$ *that is continuous and continuously differentiable on* $[0, T]$*,* $u(t) \in D(A)$ *for* $t \in (0, T)$*, and satisfies equation (5.1) on* $[0, T]$*.*

Definition 5.1.2 *A* **mild solution** *to equation (5.1) is a continuous function* $u : [0, T] \to X$ *that satisfies the integral equation*

$$u(t) = e^{tA}u_0 + \int_0^t e^{(t-s)A}N(u(s))ds. \tag{5.2}$$

A classical solution is what one typically thinks of as a solution, and a mild solution is a generalization that may have less regularity. One can show that any classical solution also satisfies the integral equation (5.2). We note that it was not necessary to introduce these definitions during the discussion of linear PDEs because, if the operator is the generator of a strongly continuous semigroup, any solution to a linear PDE is a classical solution.

Not surprisingly, the type of solution to equation (5.1) whose existence can be guaranteed is dependent upon the properties possessed by the nonlinearity, N. With relatively weak assumptions on N we obtain mild solutions.

Proposition 5.1.3 *[44] Suppose that* A *is the generator of a strongly continuous semigroup on* X *and that* $N : X \to X$ *is locally Lipschitz continuous. Then for every* $u_0 \in X$ *there exists a* $T \in (0, \infty]$ *such that the initial value problem (5.1) has a unique mild solution* $u : [0, T) \to X$*. Furthermore, if* $T < \infty$*, then* $||u(t)||_X \to \infty$ *as* $t \to T$*. If* N *is uniformly Lipschitz continuous, then* $T = \infty$*.*

If we place stronger assumptions on the nonlinearity, we obtain classical solutions.

Proposition 5.1.4 *[44] Suppose that* A *is the generator of a strongly continuous semigroup on* X *and that* $N : X \to X$ *is continuously differentiable. Then the mild solution of equation (5.1) with* $u_0 \in D(A)$ *is a classical solution.*

Typically, when the linear operator is the generator of an analytic semigroup, the solutions will be classical solutions even if the nonlinearity is only Lipschitz. Formulation of this result requires the use of fractional Banach spaces [31], and so we do not state it in detail here. The basic idea is to use the linear operator, defined on the Banach space X, to define the fractional Banach spaces X^α, for $\alpha \in (0, 1)$. One is then guaranteed local existence of solutions if the nonlinearity is Lipschitz as a function from X^α into X. This is useful because, in general, the space X^α has more regularity than X itself, and, thus, this Lipschitz condition is easier to verify. In addition, the solutions themselves lie in X^α, rather than just X, which means they, too, posses more regularity. In chapters 6 and 7 it will be necessary to use some results that require the use of fractional Banach spaces. For related details, we refer the reader to the reference [31].

The above propositions are useful because they apply to general linear operators and nonlinearities. However, often one can obtain more detailed information by working directly with the PDE itself, or with the integral formulation of solutions, equation (5.2). We provide such an example, the viscous Burgers equation:

$$u_t = u_{xx} - uu_x, \quad u(x,0) = u_0(x). \tag{5.3}$$

Note that the nonlinearity is not locally Lipschitz as a function on L^2, so we can not use the above propositions to conclude that solutions exist locally in time. It turns out, however, that the nonlinearity is Lipschitz when considered as a function on an appropriate fractional Banach space, and so one may use the theory in Henry [31] to conclude that solutions exist locally in time in L^p, for any $1 \leq p < \infty$.

We can see directly that solutions exist globally in time in L^2, as well. If we compute the derivative of the L^2 norm of the solution, we find that

$$\begin{aligned}\frac{d}{dt}\frac{1}{2}\int_{\mathbb{R}} u^2(x,t)dx &= \int_{\mathbb{R}} u(x,t)u_t(x,t)dx = -\int_{\mathbb{R}} u_x^2(x,t)dx - \frac{1}{3}\int_{\mathbb{R}} \frac{d}{dx}u^3(x,t)dx \\ &= -\int_{\mathbb{R}} u_x^2(x,t)dx \leq 0,\end{aligned}$$

where we have integrated by parts in the first term. Therefore, we find that

$$||u(t)||_{L^2} \leq ||u_0||_{L^2},$$

and, hence, solutions exist globally in time in L^2. In fact, we can get even more information about solutions. We can represent solutions to Burgers equation as

$$u(t) = e^{\partial_x^2 t}u_0 - \int_0^t e^{\partial_x^2 (t-s)}u(s)u_x(s)ds, \tag{5.4}$$

where the linear semigroup is given explicitly by

$$(e^{\partial_x^2 t}w)(x,t) = \frac{1}{\sqrt{4\pi t}}\int_{\mathbb{R}} e^{-\frac{(x-y)^2}{4t}}w(y)dy. \tag{5.5}$$

Using this formula, we may estimate

$$||e^{\partial_x^2 t}u_0||_{L^p} \leq \frac{C_p}{t^{\frac{1}{2}-\frac{1}{2p}}}||u_0||_{L^1},$$

and

$$||e^{\partial_x^2(t-s)}u(s)u_x(s)||_{L^p} \leq C_p||\frac{z}{(t-s)^{\frac{3}{2}}}e^{-\frac{z^2}{4(t-s)}}||_{L^q}||u^2(s)||_{L^r}$$
$$\leq \frac{C_p}{(t-s)^{\frac{p+1}{2p}}}||u(s)||_{L^p}^2, \tag{5.6}$$

where we have used the bound $||G*f||_{L^p} \leq ||G||_{L^q}||f||_{L^r}$ if $\frac{1}{p}+1 = \frac{1}{q}+\frac{1}{r}$ in both estimates. Next, define

$$|||u||| = \sup_{0\leq t\leq T} t^{\frac{1}{2}-\frac{1}{2p}}||u(t)||_{L^p}, \tag{5.7}$$

where T is the maximal time for which $|||u||| \leq \frac{1}{2C_p\beta(\frac{p-1}{2p},\frac{1}{p})}$, and β is the beta function defined by

$$\beta(x,y) = \int_0^1 (1-t)^{x-1}t^{y-1}dt, \text{ for } x,y>0.$$

Using equation (5.4), we may estimate

$$|||u||| \leq C_p||u_0||_{L^1} + |||u|||^2 \sup_{0\leq t\leq T}\left(t^{\frac{1}{2}-\frac{1}{2p}}\int_0^t \frac{C_p}{(t-s)^{\frac{p+1}{2p}}s^{1-\frac{1}{p}}}ds\right)$$
$$\leq C_p||u_0||_{L^1} + C_p|||u|||^2\int_0^1 \frac{1}{z^{1-\frac{1}{p}}(1-z)^{\frac{p+1}{2p}}}dz$$
$$\leq C_p||u_0||_{L^1} + C_p\beta(\frac{p-1}{2p},\frac{1}{p})|||u|||^2,$$

if $p>1$. Hence, we can estimate

$$|||u||| \leq \frac{C_p||u_0||_{L^1}}{1-C_p\beta(\frac{p-1}{2p},\frac{1}{p})|||u|||} \leq 2C_p||u_0||_{L^1}.$$

If $||u_0||_{L^1} \leq 1/(4C_p^2\beta(\frac{p-1}{2p},\frac{1}{p}))$, then the bound on $|||u|||$ is independent of T. Hence, the bound must hold for all time, and we have that

$$||u(t)||_{L^p} \leq \frac{C}{t^{\frac{1}{2}-\frac{1}{2p}}}||u_0||_{L^1} \tag{5.8}$$

for $p > 1$ if the initial data is sufficiently small.

This example demonstrates that, in some cases, one can obtain detailed information about the behavior of solutions by working directly with the integral formulation of solutions and using specific properties of the linear semigroup and nonlinearity. Of particular importance for this example was the estimate in equation (5.6), which provides information about how the semigroup interacts with the specific nonlinearity in the equation.

5.2 Invariant Manifolds

In this section, we briefly state some results related to the existence of invariant manifolds in infinite dimensional Banach spaces. We focus primarily on the case when $\sigma_u(A) = \varnothing$ and state the theorems related to the existence of a center manifold. The results of this section follow those of [10]. Other results on center manifolds can be found in, for example, [2], [9], [23] and [31].

We remark that the center manifold theorem that we will state and discuss in this section looks quite different than theorem 1.1.3, the center manifold theorem in the ODE case. The reason for this is the following. There are several versions of the center manifold theorem for PDEs, all with slightly different assumptions on both the linear operator and the nonlinearity. The infinite-dimensional center manifold theorems that are most similar to theorem 1.1.3 (see [9], for example) have relatively stringent assumptions on the nonlinearity and, as a result, will not be applicable to the examples in chapter 6. In particular, they require that the nonlinearity be C^1 in the Banach space X. We will be interested in nonlinearities of the form $N(u) = u|u|^{p-1}$, for example, which are not even continuous as functions on spaces such as $X = L^2$.

However, we will be able to apply the center manifold theorem given in [10]. The reason is that this theorem allows one to work with the integral form of solutions given in (5.2). As we saw in the example in section 5.1, working with the integral form of solutions can allow one to take advantage of certain properties that the linear semigroup has when interacting with the nonlinearity. If one works with the equation directly, then the nonlinearity must satisfy certain assumptions that are independent of its interactions with the linear operator.

In order to state the hypotheses of the theorem, we first state some notation and definitions. Write the solution given in (5.2) as $u(x,t) = \Phi_t(u_0)$, where Φ_t denotes the flow acting on the initial data u_0. We will refer to $\Phi_t : X \to X$ for $t \geq 0$ as the semigroup associated to the full, nonlinear flow. Note that, for any $t > 0$, it is possible to decompose the flow as

$$\begin{aligned} \Phi_t(u_0) &= e^{tA}u_0 + \int_0^t e^{(t-s)A}N(u(s))ds \\ &\equiv L(u_0) + R(u). \end{aligned} \tag{5.9}$$

Here, L denotes the linear part of the semigroup and R the nonlinear part. Finally, we say that a map $f : Y \to Z$ is globally Lipschitz if

$$\mathrm{Lip}(f) \equiv \sup_{y,y' \in Y,\ y \neq y'} \frac{||f(y) - f(y')||_Z}{||y - y'||_Y}$$

is finite.

Suppose the following hypotheses, which will be explained below, are satisfied [10]:

1. $\Phi_t(u_0)$ is continuous in $(t, u_0) \in \mathbb{R}^+ \times X$, and there exists a constant $q > 0$ such that

$$\sup_{0 \leq t \leq q} \mathrm{Lip}(\Phi_t) = D < \infty.$$

2. There exists a $\tau \in (0, q]$ such that Φ_τ can be decomposed as $\Phi_\tau = L + R$ where $L : X \to X$ is a bounded linear operator and $R : X \to X$ is a globally Lipschitz map.

3. There are subspaces X_i, $i = 1, 2$, of X and continuous projections $P_i : X \to X_i$ such that $P_1 + P_2 = \mathbf{1}$, $X = X_1 \oplus X_2$, L leaves X_1 and X_2 invariant, and L commutes with P_i, $i = 1, 2$. Denote by $L_i : X_i \to X_i$ the restriction of L to X_i. L_1 has a bounded inverse and there exist constants $\alpha_1 > \alpha_2 \geq 0$, $C_1 \geq 1$, $C_2 \geq 1$, such that

$$\begin{aligned} ||L_1^{-k} P_1||_X &\leq C_1 \alpha_1^{-k} \\ ||L_2^k P_2||_X &\leq C_2 \alpha_2^k, \end{aligned} \tag{5.10}$$

   where $k \geq 0$.

4. L and R satisfy

$$\frac{(\sqrt{C_1} + \sqrt{C_2})^2}{\alpha_1 - \alpha_2} \mathrm{Lip}(R) < 1.$$

The first hypothesis states that the nonlinear flow is defined for all $t \geq 0$ and all $u_0 \in X$, and that it is globally Lipschitz, uniformly in t for $0 \leq t \leq q$. The second hypothesis states that, for some fixed $t = \tau$, the flow can be decomposed into a linear and nonlinear component, so that the linear part is bounded and the nonlinear part is globally Lipschitz. Thus, we will think of the flow at this fixed τ as a map. As a result, the third hypothesis states that the linear map L can be used to decompose the space X into invariant subspaces. If $1 > \alpha_1 > \alpha_2 \geq 0$ with α_1 close to 1, then X_1 is the center-unstable subspace and X_2 is the stable subspace. Finally, the fourth hypothesis relates the size of the spectral gap, $\alpha_1 - \alpha_2$, to the size of the Lipschitz constant $\mathrm{Lip}(R)$ and the constants C_1 and C_2. If there was no spectral gap, $\alpha_1 - \alpha_2 = 0$, then this hypothesis would necessarily fail.

These hypothesis imply that one can find γ_i, $i = 1, 2$, $\alpha_2 < \gamma_2 < \gamma_1 < \alpha_1$, that satisfy the following. If we define

$$\lambda(\gamma) \equiv \frac{C_1}{\alpha_1 - \gamma} + \frac{C_2}{\alpha_2 - \gamma},$$

for $\gamma \in (\alpha_1, \alpha_2)$, then $\mathrm{Lip}(R)\lambda(\gamma_1) = \mathrm{Lip}(R)\lambda(\gamma_2) = 1$ and $\mathrm{Lip}(R)\lambda(\gamma) < 1$ for all $\gamma \in (\gamma_1, \gamma_2)$. Here, $\gamma_{1,2}$ should be understood as the nonlinear analogues of the linear growth bounds $\alpha_{1,2}$. If $\mathrm{Lip}(R) = 0$, we may take $\gamma_{1,2} = \alpha_{1,2}$.

Lastly, for any $u_0 \in X$, $\{\Phi_t(u_0)\}_{t\geq 0}$ is referred to as the positive semiorbit through u_0, and a function $u : (-\infty, 0] \to X$ is referred to as a negative semiorbit of $\{\Phi_t\}_{t\geq 0}$ if $\Phi_t(u(s)) = u(t + s)$ for any $t \geq 0$ and $s \leq -t$. The term "semiorbit" refers to the fact that one may allow time to evolve only in one direction. In general, for a linear operator with spectrum satisfying $\inf \mathrm{Re}\sigma = -\infty$, the associated PDE will have positive semiorbits only, because solutions may become unbounded in arbitrarily small, negative time. For linear operators with spectrum satisfying $\sup \mathrm{Re}\sigma = +\infty$, the associated PDE will possess only negative semiorbits.

We now state the theorem:

Theorem 5.2.1 *[10] Suppose that hypotheses 1)-4) are satisfied and $R(0) = 0$. Then there is a globally Lipschitz map $g : X_1 \to X_2$ with $g(0) = 0$ such that the Lipschitz submanifold*

$$G = \{u_1 + g(u_1) : u_1 \in X_1\}$$

of X satisfies the following properties.

i) (Invariance) The restriction on G of the semiflow $\{\Phi_t\}_{t\geq 0}$ can be extended to a Lipschitz flow on G. In particular, $\Phi_t G = G$ for all $t \geq 0$ and for any $u_0 \in G$ there exists a unique negative semiorbit $\{u(t)\}_{t\leq 0}$ in G with $u(0) = u_0$.

ii) (Lyapunov exponent) If a negative semiorbit $\{u(t)\}_{t\leq 0}$ is contained in G, then

$$\limsup_{t\to-\infty} \frac{1}{|t|} \log |u(t)| \leq -\frac{1}{\tau} \log \gamma_1,$$

where it is understood that $\log(0) = -\infty$. Conversely, if a negative semiorbit $\{u(t)\}_{t\leq 0} \subset X$ satisfies

$$\limsup_{t\to-\infty} \frac{1}{|t|} \log |u(t)| \leq -\frac{1}{\tau} \log \gamma_2,$$

then it lies on G.

iii) (Invariant Foliation) There is a continuous map $h : X \times X_2 \to X_1$ such that for each $u_0 \in G$, $h(u_0, P_2 u_0) = P_1 u_0$ and the manifold $M_{u_0} = \{h(u_0, u_2) + u_2 : u_2 \in X_2\}$ passing *through* u_0 *satisfies*

$$\Phi_t(M_{u_0}) \subset M_{\Phi_t(u_0)},\ t \geq 0$$
$$M_{u_0} = \{w \in X : \limsup_{t\to\infty} \frac{1}{t} \log |\Phi_t(w) - \Phi_t(u_0)| \leq \frac{1}{\tau} \log \gamma_2\}.$$

Moreover, $h : X \times X_2 \to X_1$ *is uniformly Lipschitz in the* X_2 *direction.*

iv) (Smoothness) If the map $\Phi_t : X \to X$ *in hypothesis 1) is* C^1*, then* $g : X_1 \to X_2$ *is* C^1*. Furthermore,* $h : X \times X_2 \to X_1$ *is* C^1 *in the* X^2 *direction, which implies that* G *and* M_{u_0} *are* C^1 *manifolds for each* $u_0 \in G$*.*

Several remarks should be made about the difference between this theorem and that of the ODE case, theorem 1.1.3. First, i) implies that any solution that lies in G exists for all $t \in \mathbb{R}$, rather than just $t \geq 0$. Second, consequence ii) has to do with growth rates of solutions in backwards time. If a solution lies on the manifold G, then it cannot grow in backwards time faster than the rate determined by γ_1. Similarly, if a solution grows in backwards time with a rate less than that given by γ_2, then it must be in the manifold G. This is related to the decay rate of solutions that are not in the manifold G. Finally, consequence iii) gives a foliation for the phase space of solutions. In addition, it implies that for each initial data not in G, the corresponding solution asymptotically approaches a solution that lies in G. Hence, the behavior of solutions in the invariant manifold governs the behavior of all solutions for large time, and, thus, stability may be determined by considering only those solutions that lie in the invariant manifold.

The main point is that, in many ways, this spectral scenario for A is the same as the ODE case in which the matrix A possesses eigenvalues with zero real part. Although stability can not be determined from the linear operator alone, one can construct a center manifold and compute the flow within it. In this manner, information regarding the nonlinearity is used to determine the stability of the zero solution. We will see examples of such a calculation in chapter 6 below. For additional examples, see [9] and [31].

Chapter 6

Examples: effect of the nonlinearity in the absence of a spectral gap

Due to the results of the previous chapter, we can now determine the stability of the zero solution to a nonlinear PDE in the following three cases. If $\sigma_u \neq 0$, then the solution is unstable. If $\sup \mathrm{Re}\sigma_s \leq -\delta < 0$ and $\sigma_u = \sigma_c = \varnothing$, then the solution is stable. If $\sup \mathrm{Re}\sigma_s \leq -\delta < 0$, $\sigma_u = \varnothing$, and there are finitely many elements of σ_c, then a center manifold reduction may be used to determine stability. In this case, the nonlinearity plays an important role in determining the behavior of solutions.

If $\sigma_u = \varnothing$, $\sup \mathrm{Re}\sigma_s = 0$, and either $\sigma_c = \varnothing$ or σ_c has finitely many elements, then there are, in general, no ways to determine the effect of the nonlinearity or the stability of the zero solution. However, one can make this determination for specific examples. We will consider three such examples in this chapter.

We remark that, in these examples, one must not only have a detailed understanding of the linear operator, but also information about how that linear operator interacts with the specific nonlinearity in each equation. Knowing information about the linear operator and nonlinearity separately is typically not enough.

6.1 Example 1: $u_t = u_{xx} - u|u|^{p-1}$

In section 3.1, we saw how both the renormalization group method and scaling variables could be used to analyze the stability of the zero solution to the heat equation, and also to determine the asymptotic (in time) form of solutions. In this section, we consider the equation

$$u_t = u_{xx} - u|u|^{p-1}, \quad u(0) = u_0, \tag{6.1}$$

where $2 < p < 4$, and use these same two techniques to investigate the effect of adding a polynomial nonlinearity to the heat equation.

First, we describe how equation (6.1) may be analyzed using the method of renormalization groups. As this analysis is well known, we focus on the main ideas involved. The presentation follows that of [5] and [6], in which more details may be found.

In order to develop some intuition for the role of the nonlinearity, consider the equation

$$u_t = u_{xx} + F(u, u_x, u_{xx}), \tag{6.2}$$

where F is a general nonlinear term. Recall that, when analyzing the linear heat equation with the renormalization group method, we looked for an appropriately defined map, R_L, the renormalization group (RG) map, which had as a stable fixed point the scale invariant solution of the heat equation. For the nonlinear equation the idea is similar. The main difference results from the fact that equation (6.2) is not necessarily scale invariant. By this we mean that, if we define

$$\begin{aligned} u_L(x,t) &= Lu(Lx, L^2 t) \\ (R_{L,F}u_0)(x) &= u_L(x,1) \end{aligned} \tag{6.3}$$

as in the linear case, equation (6.2) is not satisfied by u_L. Instead, u_L satisfies

$$\partial_t u_L = \partial_x^2 u_L + F_L(u_L, \partial_x u_L, \partial_x^2 u_L),$$

where $F_L(a,b,c) = L^3 F(L^{-1}a, L^{-2}b, L^{-3}c)$. This suggests that we must look for functions F^* and u^* so that

$$F_{L^n} \to F^*, \quad R_{L^n,F}u_0 \to u^*, \quad R_{L,F^*}u^* = u^*.$$

In other words, we need to find functions F^* and u^* so that the behavior of solutions to equation (6.2) is governed by the scale invariant solution to the equation

$$u_t = u_{xx} + F^*(u, u_x, u_{xx}),$$

which is given by the function u^*.

Suppose the nonlinearity F is simply the monomial $F(a,b,c) = a^{n_1}b^{n_2}c^{n_3}$. Then we have that $F_L = L^{3-n_1-2n_2-3n_3}F$. If $3 - n_1 - 2n_2 - 3n_3 < 0$, then, as we iterate the map, $F_{L^n} \to 0$ (because we can chose $L > 1$). Hence, we expect that $F^* \equiv 0$. This means that the behavior of solutions to equation (6.2) will be governed by the scale invariant solution to the heat equation. In other words, their behavior is the same as in the linear case. For this reason, nonlinearities satisfying $3-n_1-2n_2-3n_3 < 0$ are called "irrelevant." If instead $3 - n_1 - 2n_2 - 3n_3 \geq 0$, the nonlinearity will have a nontrivial effect on the asymptotic (in time) behavior of solutions. If $3 - n_1 - 2n_2 - 3n_3 > 0$, the nonlinearity is referred to as "relevant", and the borderline case $3 - n_1 - 2n_2 - 3n_3 = 0$ is called marginal.

Consider now equation (6.1) with $3 < p < 4$, which is the case when the nonlinearity $F = u^p$ is irrelevant. As in the linear case, take the initial time to be $t = 1$. Using the variation of constants formula, we may write the solution to (6.1) as

$$u(t) = e^{(t-1)\partial_x^2}u_0 - \int_0^{t-1} e^{((t-1)-s)\partial_x^2}u(s)|u(s)|^{p-1}ds. \tag{6.4}$$

Working in the same function space used as in the linear case,

$$Y = \{u : ||u|| = \sup(1 + k^4)(|\hat{u}(k)| + |\hat{u}'(k)|) < \infty\},$$

and using techniques similar to those illustrated in section 5.1, one can show that solutions exist and are bounded in norm for $t \in [1, L^2]$ for any fixed $L > 0$, if the initial data is sufficiently small [6]. In addition, it can be shown that the solution at time $t = L^2$ may be written as

$$u(L^2) = e^{(L^2-1)\partial_x^2} u_0 + v,$$

where $||v|| \leq C_{L,p}||u_0||^2$ (*i.e.* the constant C depends on the choice of L and p). Now, write the initial data as

$$u_0 = B_0 u^* + g_0,$$

where $B_0 = \hat{u}_0(0)$ and $\hat{u}^*(k) = e^{-k^2}$, the scale invariant solution to the heat equation, defined in equation (3.12). Because $\hat{u}^*(0) = 1$, we have that $\hat{g}_0(0) = 0$. Also, we have that $||g_0|| = ||u_0 - \hat{u}_0(0)u^*|| \leq C||u_0||$. We may now compute

$$R_{L,F}u_0 = Lu(L\cdot, L^2) = R_{L,0}u_0 + Lv(L\cdot, L^2) = B_1 u^* + g_1,$$

where $R_{L,0}$ is the RG map for the linear equation with $F = 0$, and

$$B_1 = B_0 + \hat{v}(0), \quad g_1 = R_{L,0}g_0 + Lv(L\cdot, L^2) - \hat{v}(0)u^*.$$

This implies that $\hat{g}_1(0) = 0$ and so, using the fact that $||v|| \leq C_{F,L}||u_0||^2$ we then may estimate

$$|B_1 - B_0| \leq C_{L,F}||u_0||^2$$
$$||Lv(L\cdot, L^2) - \hat{v}(0)u^*|| \leq C_{L,F}||u_0||^2.$$

Here we have used the fact that, as shown in equation (3.13), $||R_{L,0}g_0|| \leq CL^{-1}||g_0||$. Therefore, for any fixed $\delta \in (0, 1)$ we have that

$$||g_1|| \leq CL^{-1}||g_0|| + C_{L,F}||u_0||^2 \leq L^{-(1-\delta)}||u_0||$$

if $||u_0|| \leq \epsilon$, where L is chosen sufficiently large so that $CL^{-\delta} \leq 1/2$ and ϵ is chosen sufficiently small so that $C_{L,F}L^{1-\delta}\epsilon \leq 1/2$.

The idea is to now iterate this procedure. Let $v_0 = v$, and define

$$\begin{aligned} u_n &= R_{L^n,F} u_0 = B_n u^* + g_n \\ B_{n+1} &= B_n + \hat{v}_n(0) \\ g_{n+1} &= R_{L,0} g_n + L v_n L \cdot, L^2 - \hat{v}_n(0) u^*. \end{aligned}$$

The argument proceeds by induction. Assume that $|B_n| \leq (C - L^{-n})||u_0||$ and $||g_n|| \leq CL^{-(1-\delta)n}||u_0||$, which implies that $||u_n|| \leq C||u_0||$. Using the fact that $F_L = L^{-(p-3)}F$, this assumption implies that

$$\begin{aligned} ||v_n|| &\leq C_{L,F} L^{-n(p-3)} ||u_0||^2 \\ |B_{n+1} - B_n| &\leq C_{L,F} L^{-n(p-3)} ||u_0||^2 \\ ||g_{n+1}|| &\leq C L^{-1} ||g_n|| + C_{L,F} L^{-n(p-3)} ||u_0||^2. \end{aligned}$$

Because we have assumed that $p > 3$, these estimates imply convergence of the iteration scheme. In particular, there exists some B such that $B_n \to B$, and $g_n \to 0$. Using the fact that $u(x,t) = t^{-\frac{1}{2}}(R_{L^n,F} u_0)(xt^{-\frac{1}{2}})$, we then have that

$$u(x,t) \sim \frac{B}{\sqrt{t}} e^{-\frac{x^2}{4t}}. \tag{6.5}$$

Note the similarity of this equation with equation (3.15). Thus, we see that when the polynomial nonlinearity is irrelevant, $3 < p < 4$, it does not significantly affect the large time behavior of solutions.

We now turn briefly to the case when the nonlinearity is relevant, $2 < p < 3$. We have already seen, at least on a heuristic level, why the large time asymptotics of solutions might be different in this case. In order to better understand this phenomenon, notice that equation (6.1) is invariant under the scaling transformation $u_L(x,t) = L^{\frac{2}{p-1}} u(Lx, L^2 t)$. This suggests (see equation (3.14)) that we look for a scale invariant solution u^* defined by the equation

$$u(x,t) = t^{-\frac{1}{p-1}} u^*(\frac{x}{\sqrt{t}}).$$

If we insert this expression into equation (6.1), we see that u^* must satisfy

$$\partial_z^2 u^* + \frac{1}{2} z \partial_z u^* + \frac{1}{p-1} u^* - u^* |u^*|^{p-1} = 0, \tag{6.6}$$

where $z = \frac{x}{\sqrt{t}}$. It turns out that, for $2 < p < 3$, this equation has two solutions, which we will denote by u_1^* and u_2^*. It can be shown [6] that u_1^* is everywhere positive and has almost Gaussian decay at infinity. The function u_2^* can be shown to decay at infinity like $|z|^{-\frac{2}{p-1}}$.

Although the details are a bit more complex in this relevant case, these two functions can both be viewed as fixed points of an RG map. In addition, both will be stable for certain types of initial data (*i.e.* they will have different, nonempty basins of attraction). Because both u_1^* and u_2^* differ in their behavior from the function $\hat{u}^*(k) = e^{-k^2}$, which governs the behavior of solutions in the case when $p > 3$, this result shows that the behavior of solutions can be affected in a nontrivial way by the nonlinearity. Note, however, that in this case the zero solution remains stable for any $2 < p < 4$, it's just that the asymptotic (in time) form of solutions changes.

Finally, we remark that if one considers the equation

$$u_t = u_{xx} + u|u|^{p-1},$$

for any $p > 1$, then there exist solutions that blow up in finite time [22], [38]. In the case where $2 < p < 3$, the zero solution is, in fact, unstable.

We now turn to the analysis of equation (6.1) using invariant manifolds, which was first presented in [58]. In order to apply the center manifold theorem to the scaled version of equation (6.1), we will need to use some properties of the original solution, which we now state.

Proposition 6.1.1 *For initial data satisfying* $u_0 \in L^1$ *with sufficiently small norm, there exists a* $T > 0$ *such that the corresponding solution to equation (6.1) satisfies*

$$||u(t)||_{L^\infty} \leq \frac{C}{t^{\frac{1}{2}}}||u_0||_{L^1} \tag{6.7}$$

for all $0 < t < T$*. Furthermore, if* $u_0 \in L^2$*, then the solution to equation (6.1) satisfies*

$$||u(t)||_{L^2} \leq ||u_0||_{L^2}, \tag{6.8}$$

for all $t \geq 0$

Proof Local existence can be obtained by working in fractional Banach spaces X^α, with $X = L^q$ for $1 \leq q < \infty$. To obtain the estimate in equation (6.8), we compute

$$\frac{d}{dt}\frac{1}{2}\int u^2(x,t)dx = -\int u_x^2(x,t)dx - \int u^2(x,t)|u(x,t)|^{p-1}dx \leq 0.$$

We note that the boundary terms that result from integrating by parts vanish because $u(t) \in X^\alpha$ for $0 < \alpha < 1$ locally in time and, hence, has the necessary decay at infinity.

To obtain the estimate in equation (6.7), we use the integral formulation of solutions to bound

$$||u(t)||_\infty \leq \frac{1}{\sqrt{4\pi t}}||u_0||_{L^1} + \int_0^t ||e^{\partial_x^2(t-s)}u(s)|u(s)|^{p-1}||_\infty ds.$$

If $3 \leq p < 4$, we use the fact that

$$\begin{aligned} ||e^{\partial_x^2(t-s)}u(s)|u(s)|^{p-1}||_\infty &\leq \frac{1}{\sqrt{4\pi(t-s)}}||u(s)|u(s)|^{p-1}||_{L^1} \\ &\leq \frac{1}{\sqrt{4\pi(t-s)}}||u(s)||_\infty^{p-2}||u(s)||_{L^2}^2. \end{aligned}$$

Define $|||u||| = \sup_{0\leq t\leq T} t^{\frac{1}{2}}||u(t)||_\infty$. We then have that

$$\begin{aligned} |||u||| &\leq \frac{1}{\sqrt{4\pi}}||u_0||_{L^1} + ||u_0||_{L^2}^2|||u|||^{p-2}\left(\sup_{0\leq t\leq T} t^{\frac{1}{2}}\int_0^t \frac{1}{\sqrt{4\pi(t-s)}s^{\frac{p-2}{2}}}ds\right) \\ &\leq \frac{1}{\sqrt{4\pi}}||u_0||_{L^1} + \frac{1}{\sqrt{4\pi}}||u_0||_{L^2}^2\beta(\frac{1}{2},\frac{4-p}{2})T^{\frac{4-p}{2}}|||u|||^{p-2} \\ &\equiv \frac{1}{\sqrt{4\pi}}||u_0||_{L^1} + M(T)|||u|||^{p-2}. \end{aligned}$$

Hence, if T is chosen to be the maximal time such that

$$|||u||| \leq \left(\frac{1}{2M(T)}\right)^{\frac{1}{p-3}},$$

(or T is chosen sufficiently small so that $M(T) \leq 1/2$, if $p = 3$) then we see that

$$|||u||| \leq \frac{||u_0||_{L^1}}{1 - M(T)|||u|||^{p-3}} \leq 2||u_0||_{L^1}.$$

Therefore, if the initial data is sufficiently small, *i.e.* satisfies $||u_0||_{L^1} \leq \sqrt{\pi}\,(1/2M(T))^{\frac{1}{p-3}}$, then the result holds for all $0 < t < T$. The case in which $2 < p < 3$ may be proved similarly by noting that

$$||e^{\partial_x^2(t-s)}u(s)|u(s)|^{p-1}||_{L^\infty} \leq \frac{C}{(t-s)^{\frac{1}{4}}}||u(s)||_{L^\infty}^{p-1}||u(s)||_{L^2}.$$

□

Define the scaling transformation

$$u(x,t) = (t+1)^{-\frac{1}{p-1}} w(\frac{x}{\sqrt{t+1}}, \log(t+1))$$
$$\eta = \frac{x}{\sqrt{t+1}}, \quad \tau = \log(t+1). \tag{6.9}$$

Inserting this expression into equation (6.1), we see that

$$\partial_\tau w = \mathcal{L} w + \left(\frac{3-p}{2(p-1)}\right) w - w|w|^{p-1}, \tag{6.10}$$

where $\mathcal{L} = \partial_\eta^2 + \frac{1}{2}\eta\partial_\eta + \frac{1}{2}$ is the linear operator studied in section 3.1.2. Recall the spectrum of $\mathcal{L}$, given in proposition 3.1.3. We will choose $m = m(p)$ such that $\sigma_c = \{\frac{3-p}{2(p-1)}\}$ and $\sigma_s = \sigma(\mathcal{L}) \setminus \sigma_c$. Note that the additional linear term in equation (6.10) has shifted the spectrum to the right by the amount $\frac{3-p}{2(p-1)}$. Hence, σ_c is not truly the "center" part of the spectrum, unless $p = 3$. However, it is close to the imaginary axis for $2 < p < 4$, and so we will still refer to in this manner.

In order to apply the center manifold theorem, theorem 5.2.1, we will need to verify the hypotheses 1) - 4) given in section 5.2. To that end, write the solution equation (6.10) as

$$w(\tau) = e^{\mathcal{L}\tau} w_0 - \int_0^\tau e^{\mathcal{L}(\tau-s)} w(s)|w(s)|^{p-1} ds. \tag{6.11}$$

In addition, we note that the following proposition regarding the linear semigroup was proven in [25], proposition A.5.

Proposition 6.1.2 *[25] Let $1 \le p \le q \le \infty$, $m \ge 0$, and $T > 0$. Then for any $\alpha \in \mathbb{N}$ there exists a constant C such that*

$$||b^m \partial^\alpha (e^{\mathcal{L}\tau} f)||_{L^p} \le \frac{C}{a(\tau)^{\frac{1}{2}(\frac{1}{q}-\frac{1}{p})+\frac{\alpha}{2}}} ||b^m f||_{L^q}, \tag{6.12}$$

where $b(\eta) = (1+\eta^2)^{\frac{1}{2}}$ and $a(\tau) = 1 - e^{-\tau}$.

We next prove a proposition regarding the nonlinear part of the solution.

Proposition 6.1.3 *Fix $T > 0$ and m. For any $w \in C^0([0,T], L^2(m))$, define*

$$R(\tau) = \int_0^\tau e^{\mathcal{L}(\tau-s)} w(s)|w(s)|^{p-1} ds. \tag{6.13}$$

Then $R(\tau) \in C^0([0,T], L^2(m))$ and there exists a $C(m, r_0, T)$ such that, if w_1, $w_2 \in C^0([0,T], L^2(m))$ with $\sup_{0\le\tau\le T} ||w_i(\tau)||_{L^2(m)} \le r_0$, then the corresponding integral terms *satisfy*

$$\sup_{0\le\tau\le T}||R_1(\tau)-R_2(\tau)||_{L^2(m)} \le C(m,T,r_0)\sup_{0\le\tau\le T}||w_1(\tau)-w_2(\tau)||_{L^2(m)}.$$

Furthermore, the constant $C(m,T,r_0)\to 0$ *as* $T\to 0$ *and as* $r_0\to 0$.

Proof Using the bound given in equation (6.12), we may estimate

$$\begin{aligned}||R_1(\tau)-R_2(\tau)||_{L^2(m)} &\le \int_0^\tau ||e^{\mathcal{L}(\tau-s)}w(s)|w(s)|^{p-1}||_{L^2(m)}ds\\ &\le \int_0^\tau \frac{C}{a(\tau-s)^{\frac{1}{2}(1-\frac{1}{2})}}|||w_1(s)|^{p-1}(w_1(s)-w_2(s))||_{L^1(m)}ds\\ &\quad+\int_0^\tau \frac{C}{a(\tau-s)^{\frac{1}{2}(1-\frac{1}{2})}}||w_2(s)(|w_1(s)|^{p-1}-|w_2(s)|^{p-1})||_{L^1(m)}ds,\end{aligned}$$

where $L^1(m)=\{w:(1+\eta^2)^{\frac{m}{2}}w(\eta)\in L^1\}$. To estimate the first term on the right hand side, we use the fact that

$$\begin{aligned}&\int(1+\eta^2)^{\frac{m}{2}}|w_1|^{p-1}|w_1-w_2|d\eta\\ &\quad\le\left(\int(1+\eta^2)^m|w_1-w_2|^2d\eta\right)^{\frac{1}{2}}\left(\int(1+\eta^2)^m|w_1|^2\frac{1}{(1+\eta^2)^m}|w_1|^{(2(p-1)-2)}d\eta\right)^{\frac{1}{2}}\\ &\quad\le||w_1(s)||_\infty^{p-2}||w_1(s)||_{L^2(m)}||w_1(s)-w_2(s)||_{L^2(m)}.\end{aligned}$$

To estimate the second term, we define $f(\sigma)=|\sigma w_1+(1-\sigma)w_2|^{p-1}$ to obtain

$$\begin{aligned}||w_2(s)(|w_1(s)|^{p-1}-|w_2(s)|^{p-1})||_{L^1(m)} &= ||w_2(s)\int_0^1 f'(\sigma)d\sigma||_{L^1(m)}\\ &\le C\left(\Sigma_{n=0}^{p-2}||w_1(s)||_\infty^n||w_2(s)||_\infty^{p-2-n}\right)||w_2(s)(w_1(s)-w_2(s))||_{L^1(m)}\\ &\le C\left(\Sigma_{n=0}^{p-2}||w_1(s)||_\infty^n||w_2(s)||_\infty^{p-2-n}\right)||w_2(s)||_{L^2(m)}||w_1(s)-w_2(s)||_{L^2(m)}.\end{aligned}$$

Next, note that the estimate (6.7) can be transformed, when written in terms of the scaling variables, into a bound for w. For all $0\le\tau\le T$, we have that

$$||w(\tau)||\le\frac{C(T)}{(e^\tau-1)^{\frac{1}{2}}}||w_0||_{L^2(m)}. \tag{6.14}$$

Hence, we have that

$$\sup_{0\le\tau\le T} ||R_1(\tau)-R_2(\tau)||_{L^2(m)} \le$$
$$C(m,r_0,T)\left(\sup_{0\le\tau\le T}\int_0^\tau \frac{C}{a(\tau-s)^{\frac{1}{4}}(e^s-1)^{\frac{p-2}{2}}}ds\right)\sup_{0\le\tau\le T}||w_1(\tau)-w_2(\tau)||_{L^2(m)}, \tag{6.15}$$

which proves the result. □

We note that this proposition may be used to show that R is differentiable at 0, with $DR(0)=0$. To see this, we must show that, given any $\epsilon>0$, r_0 can be chosen sufficiently small so that

$$\frac{||R(w(\tau))-R(0)-DR(0)||_{L^2(m)}}{||w(\tau)||_{L^2(m)}} < \epsilon$$

uniformly for $0\le\tau\le T$. This can be done using the bound in equation (6.15).

Next, we use proposition 6.1.3 to prove that solutions to equation (6.10) exist for all $\tau\ge 0$.

Proposition 6.1.4 *Given any sufficiently small $w_0\in L^2(m)$, there exists a solution to equation (6.10) satisfying $w(\tau)\in C^0([0,\infty),L^2(m))$.*

Proof Using the estimate in the above lemma and a fixed point argument, one can show that solutions exist locally in time. To see that they exist globally, as well, we compute

$$\begin{aligned}\frac{d}{d\tau}\frac{1}{2}\int \eta^{2m}w^2(\eta,\tau)d\eta &= -\int 2m\eta^{2m-1}w_\eta w - \int \eta^{2m}w_\eta^2 \\ &\quad -\left(\frac{2m+1}{4}-\frac{1}{2}-\frac{3-p}{2(p-1)}\right)\int \eta^{2m}w^2 - \int \eta^{2m}w^2|w|^{p-1} \\ &\le -\left(\frac{2m+1}{4}-\frac{1}{2}-\frac{3-p}{2(p-1)}-\epsilon\right)\int \eta^{2m}w^2 + C_\epsilon\int w^2.\end{aligned}$$

In the last inequality, we have used the fact that

$$m(2m-1)\int \eta^{2m-2}w^2 \le m(2m-1)\int\left(\epsilon\eta^{2m}w^2+C_\epsilon w^2\right).$$

Next, note that the estimate (6.8) implies that

$$||w(\tau)||_{L^2} \le e^{\frac{5-p}{4(p-1)}\tau}||w_0||_{L^2}.$$

Hence, we see that

$$\frac{d}{d\tau}\frac{1}{2}\int \eta^{2m}w^2 d\eta \leq -\delta \int \eta^{2m}w^2 d\eta + C_\epsilon e^{\frac{5-p}{4(p-1)}\tau}||w_0||_{L^2},$$

where $\delta > 0$, if m is chosen so that $\frac{2m+1}{4} - \frac{1}{2} - \frac{3-p}{2(p-1)} > 0$. This result shows that, although $||w(\tau)||_{L^2(m)}$ may become unbounded as $\tau \to \infty$, it cannot do so in finite time. Hence, solutions exist globally in time. □

Proposition 6.1.3 and 6.1.4 can be used to show that hypotheses 1) - 4) of theorem 5.2.1 are satisfied for a slightly modified version of equation (6.10), in which the nonlinearity is cutoff outside of a small neighborhood of zero in $L^2(m)$. This is necessary so that the size of the Lipschitz constant for the nonlinearity can be made sufficiently small by choosing this neighborhood to be sufficiently small. To that end, let $\chi_{r_0}(w) : L^2(m) \to \mathbb{R}^+$ be a smooth function satisfying $\chi_{r_0}(w) = 1$ if $||w||_{L^2(m)} \leq r_0$ and $\chi_{r_0}(w) = 0$ if $||w||_{L^2(m)} \geq 2r_0$. We then consider the equation

$$\partial_\tau w = \mathcal{L}w + \left(\frac{3-p}{2(p-1)}\right) w - \chi_{r_0}(w)w|w|^{p-1}. \tag{6.16}$$

The previous two propositions can be modified so that they apply to the above equation, as well. Therefore, we have the following result:

Proposition 6.1.5 *Let $\Phi_1^{r_0}$ be the semiflow associated to equation (6.16) at time $\tau = 1$. Then, if $r_0 > 0$ is sufficiently small, the semigroup can be decomposed as*

$$\Phi_1^{r_0} = L + \mathcal{R},$$

where L is a bounded linear map, and $\mathcal{R}$ is a globally Lipschitz map such that $Lip(\mathcal{R}) \leq C(r_0)$ where $C(r_0) \to 0$ as $r_0 \to 0$. Furthermore, $\mathcal{R}$ is C^1 with $\mathcal{R}(0) = D\mathcal{R}(0) = 0$.

As a result, the hypotheses to theorem 5.2.1 are satisfied for equation (6.16). Hence, we know that a one dimensional center-unstable manifold exists in the phase space of equation (6.16). Now we may compute the flow on the center manifold, for $2 < p < 4$. This will be useful because it will provide an alternative perspective to the bifurcation at $p = 3$ that was illustrated using the renormalization group method.

We write the solution as $w(\eta, \tau) = w_c(\eta, \tau) + w_s(\eta, \tau)$. We have that $w_c(\tau, \eta) = \alpha(\tau)\phi_0(\eta)$, where ϕ_0 is the eigenfunction associated to the zero eigenvalue, given in equation (3.20). In addition, $w_s = g(w_c)$ for some function g that defines the center manifold.

Furthermore, we may decompose equation (6.16) as

$$\begin{aligned}\partial_\tau w_c &= \mathcal{L}_c w_c + \left(\frac{3-p}{2(p-1)}\right) w_c - P_c \chi_{r_0}(w_c+w_s)(w_c+w_s)|w_c+w_s|^{p-1} \\ \partial_\tau w_s &= \mathcal{L}_s w_s + \left(\frac{3-p}{2(p-1)}\right) w_s - P_s \chi_{r_0}(w_c+w_s)(w_c+w_s)|w_c+w_s|^{p-1},\end{aligned} \tag{6.17}$$

where P_c and $P_s = \mathbf{1} - P_c$ are the projection operators associated with the center and stable subspaces, respectively. For the operator $\mathcal{L}$, these operators have an explicit form. They are defined in terms of the Hermite polynomials [25]

$$H_j(\eta) = \frac{2^j}{j!} e^{\frac{\eta^2}{4}} \partial_\eta^j (e^{-\frac{\eta^2}{4}}), \tag{6.18}$$

which are the eigenfunctions of the adjoint operator $\mathcal{L}^* = \partial_\eta^2 - \frac{1}{2}\eta\partial_\eta$. In general, for any $n < m - \frac{1}{2}$, the projection onto the first n eigenvalues is given by

$$(P_n f)(\eta) = \sum_{j \le n} \left(\int H_j(\zeta) f(\zeta) d\zeta \right)^{\frac{1}{2}} \phi_j(\eta).$$

Hence,

$$(P_c f)(\eta) = \left(\int H_0(\zeta) f(\zeta) d\zeta \right)^{\frac{1}{2}} \phi_0(\eta). \tag{6.19}$$

We next note that $w_c(\tau, \eta) = \alpha(\tau)\phi_0(\eta)$ for some function $\alpha(\tau)$, where $\mathcal{L}\phi_0 = 0$. Furthermore, $g(w_c) = \mathcal{O}(\alpha^2)$. This last fact follows from the fact that g is C^1, and $g'(0) = 0$, which can be seen from the proof of theorem 5.2.1 [10]. We find that the flow on the center manifold is given by

$$\partial_\tau \alpha = \left(\frac{3-p}{2(p-1)}\right) \alpha - \frac{1}{p^{\frac{1}{2}} (4\pi)^{\frac{p-1}{2}}} \alpha^p + \mathcal{O}(\alpha^{p+1}). \tag{6.20}$$

Remark 6.1.6 *Notice that it was not necessary to determine the form of the function g in order to compute the leading order dynamics on the center manifold. This is due to the form of the nonlinearity and the fact that $g(\alpha) = \mathcal{O}(\alpha^2)$.*

If $2 < p < 3$, then there are two fixed points to this equation:

$$\alpha = 0, \quad \alpha = \alpha^* \equiv \sqrt{4\pi} p^{\frac{1}{2(p-1)}} \left(\frac{3-p}{2(p-1)} \right)^{\frac{1}{p-1}}.$$

One can show that α^* is stable, and the behavior of solutions will be given by $w \sim \alpha^* \phi_0 + g(\alpha^*)$ as $\tau \to \infty$. We remark that we technically need to require that p is sufficiently close to 3 so that the fixed point α^* lies within the neighborhood in which the center manifold was defined. If $p \geq 3$, then $\alpha = 0$ is the only fixed point, and it is stable. Since $g(0) = 0$, this implies that $w \sim \alpha_0 e^{-\frac{(p-3)}{2(p-1)}\tau} \phi_0$ as $\tau \to \infty$. Hence, we see that a bifurcation occurs on the center manifold when $p = 3$, indicating that the asymptotic form of solutions will also change.

By transforming back to the original variables, we can use this information to determine the asymptotic form of u. We find that

$$u(x,t) \begin{cases} = \frac{\alpha^*}{\sqrt{4\pi}(t+1)^{\frac{1}{p-1}}} e^{-\frac{x^2}{4(t+1)}} + \frac{g(\alpha^*)}{(t+1)^{\frac{1}{p-1}}} + \mathcal{O}((t+1)^{-\frac{1}{p-1} - \frac{(3-p)}{2}}) \text{ if } 2 < p < 3 \\ \sim \frac{1}{\sqrt{4\pi}(t+1)(\frac{\log(t+1)}{2\pi\sqrt{3}} + \frac{1}{\alpha_0^2})} e^{-\frac{x^2}{4(t+1)}} \text{ if } p = 3 \\ = \frac{\alpha_0}{\sqrt{4\pi(t+1)}} e^{-\frac{x^2}{4(t+1)}} + \mathcal{O}((t+1)^{-\frac{p^2-3p+2}{2(p-1)}}) \text{ if } 3 < p < 4. \end{cases} \tag{6.21}$$

Note that, when $3 < p < 4$, the leading order behavior is independent of p. This is due to the irrelevance of the nonlinearity, as explained in the context of the renormalization group method.

Thus, we see that the method of scaling variables may be used to determine the leading order asymptotics of solutions to equation (6.1) and to analyze the bifurcation that occurs for $p = 3$. In addition, by considering initial data in the weighted space $L^2(m)$ for $m > 1$, one can determine higher order terms in the asymptotic expansion.

Remark 6.1.7 *The fixed point α^* corresponds to the solution to equation (6.6) denoted by u_1^*. Thus, the above center manifold calculation proves that the basin of attraction of u_1^* includes some neighborhood of the origin in $L^2(m)$.*

6.2 Example 2: $u_t = u_{xx} + c\tanh(\frac{c}{2}x)u_x - u|u|^{p-1}$

We now turn to the analysis of the following equation,

$$u_t = u_{xx} + c\tanh(\frac{c}{2}x)u_x - u|u|^{p-1}, \quad u(x,0) = u_0(x). \tag{6.22}$$

The linear operator, $A_2 = \partial_x^2 + c\tanh(\frac{c}{2}x)\partial_x$, was studied in section 3.2, and we will use several of the results described in that section in the analysis of this nonlinear equation. As remarked in that section, this linear operator is related to that which is obtained by linearizing around the traveling front in Burgers equation. The nonlinear stability of the front in Burgers equation will be studied in section 6.3 below. We focus on equation (6.22) first because an understanding of the role that the nonlinearity u^p plays in the behavior of solutions will help us better understand the nonlinearity associated to the stability of the front.

In order to study this equation, we use the scaling variables and decomposition of the solution that was introduced in section 3.2. The key difference between the analysis of the linear equation (3.25) and the nonlinear equation (6.22) is that the nonlinearity will create some coupling between the two terms in the decomposition defined in equation (3.36). In order to deal with this coupling, we will define an additional component of the decomposition, which will be denoted by u_3. It is important that the structure of the linear operator in the equations for u_1 and u_2 be preserved, so that we may analyze them using scaling variables. Therefore, all of the coupled terms involving combinations of u_1, u_2, and u_3 will be placed into the equation of evolution for u_3.

Let u_1, u_2, and u_3 be solutions of

$$\partial_t u_1 = \partial_x^2 u_1 + c\partial_x u_1 - \frac{u_1|u_1|^{p-1}}{(1+e^{-cx})^{p-1}}, \tag{6.23}$$

$$\partial_t u_2 = \partial_x^2 u_2 - c\partial_x u_2 - \frac{u_2|u_2|^{p-1}}{(1+e^{+cx})^{p-1}}, \tag{6.24}$$

$$\partial_t u_3 = \mathcal{L}_3 u_3 - N_3(u_3) - F_3(x,t), \tag{6.25}$$

where

$$\begin{aligned}\mathcal{L}_3 u_3 + N_3(u_3) + F_3(x,t) = {} & \partial_x^2 u_3 + c\tanh(\frac{c}{2}x)\partial_x u_3 \\ & - \left(\frac{u_1}{1+e^{-cx}} + \frac{u_2}{1+e^{+cx}} + u_3\right)|\frac{u_1}{1+e^{-cx}} + \frac{u_2}{1+e^{+cx}} + u_3|^{p-1} \\ & + \left(\frac{u_1}{1+e^{-cx}}\right)|\frac{u_1}{1+e^{-cx}}|^{p-1} + \left(\frac{u_1}{1+e^{-cx}}\right)|\frac{u_1}{1+e^{-cx}}|^{p-1},\end{aligned} \tag{6.26}$$

and with initial data

$$\begin{aligned} u_{1,2}(x,0) &= u(x,0) - \operatorname{sech}(\frac{c}{2}x)u(x,0) \\ u_3(x,0) &= \operatorname{sech}(\frac{c}{2}x)u(x,0). \end{aligned} \tag{6.27}$$

We then see that

$$u(x,t) = \frac{u_1(x,t)}{1+e^{-cx}} + \frac{u_2(x,t)}{1+e^{cx}} + u_3(x,t) \tag{6.28}$$

is a solution to equation (6.22). Our ultimate goal is to understand the asymptotic behavior of u, and we will do so by determining the asymptotic behavior of u_1, u_2, and u_3. We first focus on the behavior of u_1 and u_2 and return to the analysis of u_3 below.

We state the details for u_1 only. Those of u_2 are similar. Define the scaling variables (η, τ) and the function w_1 in a manner similar to section 3.2:

$$\begin{aligned} u_1(x,t) &= \frac{1}{(t+1)^{\frac{1}{(p-1)}}} w_1\left(\frac{x+c(t+1)}{\sqrt{t+1}}, \log(t+1)\right) \\ \eta &= \frac{x+c(t+1)}{\sqrt{t+1}}, \quad \tau = \log(t+1), \end{aligned} \tag{6.29}$$

Inserting these into equation (6.23), we obtain

$$\partial_\tau w_1 = \mathcal{L} w_1 + \left(\frac{3-p}{2(p-1)}\right) w_1 - \frac{w_1|w_1|^{p-1}}{(1+e^{-c(\eta e^{\frac{\tau}{2}} - ce^{\tau})})^{p-1}} \tag{6.30}$$

where $\mathcal{L} = \partial_\eta^2 + \frac{1}{2}\eta\partial_\eta + \frac{1}{2}$.

The analysis of equation (6.30) will be similar to that of equation (6.10) in section 6.1. We would like to apply the center manifold theorem 5.2.1, however we cannot do so directly because equation (6.30) is nonautonomous. Therefore, define

$$\tau = \ln\left(\frac{2-\zeta}{\zeta}\right). \tag{6.31}$$

Equation (6.30) may then be written as

$$\begin{aligned} \partial_\tau w_1 &= \mathcal{L} w_1 + \left(\frac{3-p}{2(p-1)}\right) w_1 - \frac{w_1^p}{(1+e^{-c\gamma(\eta,\zeta)})^{p-1}} \\ \partial_\tau \zeta &= -\zeta + \frac{1}{2}\zeta^2, \end{aligned} \tag{6.32}$$

where $\gamma(\eta,\zeta) = \eta e^{\frac{\tau(\zeta)}{2}} - ce^{\tau(\zeta)}$. Since $\zeta = 0$, which corresponds to $\tau = \infty$, is stable, we study the evolution of this system near $(0,0) \in L^2(m) \times [0,1]$. Using methods analogous to those of section 6.1, we can show that theorem 5.2.1 can be applied to the integral form of solutions to equation (6.32), after cutting off the nonlinear term outside a sufficiently small neighborhood of the origin in $L^2(m)$. Thus, we may conclude that there exists a one-dimensional center-unstable manifold in the phase space, and we may represent solutions on this manifold as $(w_1(\eta,\tau),\zeta(\tau)) = \alpha(\tau)\phi_0(\eta)+g_1(\alpha)$. The dynamics on the center manifold are given by

$$\dot{\alpha} = \frac{3-p}{2(p-1)}\alpha - B_1\alpha^p + \mathcal{O}(\alpha^{p-1}), \tag{6.33}$$

where $B_1 = P_c\left(\frac{\phi_0^p}{(1+e^{-c\gamma(\eta,\zeta(\alpha))})^{p-1}}\right) > 0$. As in the previous section, we see that for $p > 3$ the fixed point $\alpha = 0$ is stable, while for $p < 3$ the fixed point $\alpha* = \left(\frac{3-p}{2(p-1)B_1}\right)^{\frac{1}{p-1}}$ is stable. Hence, the asymptotics of w_1 are the same as in the previous section, with a bifurcation occurring at $p = 3$. A similar result holds for w_2, the scaled version of u_2. Thus, in some sense the "relevance" of the polynomial nonlinearity, $u|u|^{p-1}$, for the linear operator in equation (6.22) seems to be the same as that of equation (6.1).

However, recall that we are ultimately interested in the asymptotic behavior of u_1 and u_2 in terms of the original, unscaled variables. If, for example, we use the above result to determine this behavior for u_1 with $3 < p < 4$, we obtain

$$\begin{aligned}\frac{u_1(x,t)}{1+e^{-cx}} &= \frac{\alpha_0}{\sqrt{4\pi(t+1)}(1+e^{-cx})}e^{-\frac{(x+c(t+1))^2}{4(t+1)}} + \mathcal{O}((t+1)^{-\frac{2m+1}{4}}) \\ &= \frac{\alpha_0}{\sqrt{4\pi(t+1)}}\text{sech}(\frac{c}{2}x)e^{-\frac{c^2}{4}(t+1)}e^{-\frac{x^2}{4(t+1)}} + \mathcal{O}((t+1)^{-\frac{2m+1}{4}}).\end{aligned} \tag{6.34}$$

Notice that the Gaussian, when written in terms of the (x,t) variables, actually decays exponentially in time. Thus, it is the seemingly higher order terms that control the temporal decay of u_1. As a result of the above analysis, w_1 and w_2 exhibit Gaussian-like decay for all $2 < p < 4$, and so the large time asymptotics of both u_1 and u_2 for all $2 < p < 4$ will be governed by the algebraic decay that results from the location of the essential spectrum of the linear operator in equation (6.30). Therefore, in some sense the nonlinearity $u|u|^{p-1}$ is irrelevant for all $2 < p < 4$, and the algebraic temporal decay that results from the linear operator controls the dynamics for all values of p.

We now turn to the study of the behavior of u_3. Note that the spectrum of the linear operator associated with the evolution of u_3 has essential spectrum that touches the imaginary axis at the origin. This might cause some difficulty in the analysis of the behavior of solutions. To overcome this issue, we will study the evolution of u_3 in an exponentially weighted space, as in [52]. This will push the essential spectrum off the imaginary axis.

Remark 6.2.1 *Note that, by studying the evolution of u_3 is an exponentially weighted space, we are not restricting the form of the initial data for the original u variable. This is because*

$$u(x,t) = \frac{1}{(1+e^{-cx})}u_1(x,t) + \frac{1}{(1+e^{+cx})}u_2(x,t) + u_3(x,t),$$

and, even though each component is effectively analyzed in some exponentially weighted space, their sum will still lie in L^2, for example. Intuitively, one can think of u_1 as representing the far field behavior of u near $x = +\infty$, of u_2 as representing the far field behavior of u near $x = -\infty$, and of u_3 as representing the near field behavior of u near $x = 0$.

In order to simplify the analysis, we present the details only for the specific case in which $p = 3$. Define $w_3(x, \tau) = \cosh(\frac{c}{2}x)u_3(x)$. We see that the evolution of w_3 is given by

$$\partial_t w_3 = \tilde{\mathcal{L}}_3 w_3 + \tilde{N}_3(w_3, x) + \tilde{F}_3(x, t), \tag{6.35}$$

where

$$\begin{aligned} \tilde{\mathcal{L}}_3 w_3 &= \partial_x^2 w_3 - \frac{c^2}{4} w_3 - 3\left(\frac{u_1}{1+e^{-cx}} + \frac{u_2}{1+e^{+cx}}\right)^2 w_3 \\ \tilde{N}_3(w_3, x) &= -3\mathrm{sech}(\frac{c}{2}x)\left(\frac{u_1}{1+e^{-cx}} + \frac{u_2}{1+e^{+cx}}\right) w_3^2 - \mathrm{sech}^2(\frac{c}{2}x) w_3^3 & (6.36) \\ \tilde{F}_3(x, t) &= -3\cosh(\frac{c}{2}x)\left(\frac{u_1}{1+e^{-cx}}\right)\left(\frac{u_2}{1+e^{+cx}}\right)\left(\frac{u_1}{1+e^{-cx}} + \frac{u_2}{1+e^{+cx}}\right). & (6.37) \end{aligned}$$

In order to analyze this equation, we will split the linear operator up according to

$$\begin{aligned} \tilde{\mathcal{L}}_3 w_3 &= \partial_x^2 w_3 - \frac{c^2}{4} w_3 + A(t) w_3 \\ A(t) w_3 &= -3\left(\frac{u_1}{1+e^{-cx}} + \frac{u_2}{1+e^{+cx}}\right)^2 w_3. \end{aligned} \tag{6.38}$$

Consider the integral form of solutions to equation (6.35), which can be written

$$\begin{aligned} w_3(t) = e^{(\partial_x^2 - \frac{c^2}{4})t} w_3(0) &+ \int_0^t e^{(\partial_x^2 - \frac{c^2}{4})(t-s)} A(s) w_3(s) ds + \int_0^t e^{(\partial_x^2 - \frac{c^2}{4})(t-s)} \tilde{N}_3(w_3(s)) ds \\ &+ \int_0^t e^{(\partial_x^2 - \frac{c^2}{4})(t-s)} \tilde{F}_3(s) ds. \end{aligned} \tag{6.39}$$

Using the above expression and a fixed point argument, one can show that solutions to equation (6.35) exist locally in time in L^2. (For the details of such an argument, see the proof of proposition 6.3.6 in section 6.3.)

We are interested in the asymptotic behavior of solutions to equation (6.35). The semigroup associated to the linear operator $\partial_x^2 - \frac{c^2}{4}$ decays exponentially in time, which implies that the first three terms on the right hand side of equation (6.39) will decay exponentially in time, as well. The last term, however, which results from the inhomogeneity, will only decay as fast as u_1 and u_2. As a result, the asymptotic behavior of w_3 will be the same as that of u_1 and u_2. This can be shown rigorously, as in proposition 6.3.7.

This result implies that it is really the far-field behavior of the solution to equation (6.22) that governs its large time asymptotics. We saw above that, for any $2 < p < 4$, the far-field components of the solution decay algebraically at a rate that is determined by the location of the essential spectrum of the linear operator $\mathcal{L} = \partial_\eta^2 + \frac{1}{2}\eta\partial_\eta + \frac{1}{2}$. Thus, we have that the solution to equation (6.22) satisfies

$$||u(t)||_{L^2} \leq \frac{C}{(t+1)^{\frac{2m+1}{4}}}, \tag{6.40}$$

if $u(0) \in L^2(m)$ and is sufficiently small.

6.3 Example 3: $u_t = u_{xx} + c\tanh(\frac{c}{2}x)u_x + \frac{c^2}{2}\text{sech}(\frac{c}{2}x)u - 2uu_x$, (local) stability of traveling fronts in Burgers equation

The example studied in this section is the equation that determines the stability of the traveling front in Burgers equation. It is the example that motivated much of the preceding analysis and, along with chapter 4, it is the focus of the original research contained in this dissertation. Because of this fact, we present its derivation in detail.

Burgers equation is given by

$$\partial_t U = \partial_y^2 U - \partial_y(U^2), \tag{6.41}$$

where $U = U(y,t)$, $y \in \mathbb{R}$, $t > 0$, and $U(y,0) = U_0(y)$. Burgers equation was originally introduced in the late 1930's by J. M. Burgers to model turbulent flows [7], [8]. Recently, it has been extensively studied as a prototypical model in the context of stability analysis with weighted norms [52], the application of the renormalization group method to fronts (*i.e.* non-Gaussian fixed points) [6], and the analysis of scalar conservation laws (see example 8.6 of [60]). Our main motivation for studying the stability of the traveling front solution to Burgers equation was to determine if the method of scaling variables, illustrated in section 3.1, could be applied to linear operators other than the Laplacian.

In order to study traveling wave solutions to equation (6.41), we define the moving coordinate $x = y - ct$, where $c > 0$ is the speed of the wave. We then look for a solution of the form $\varphi(y,t) = \varphi(y - ct) = \varphi_c(x)$. This solution must satisfy

$$-c\varphi_c' = \varphi_c'' - 2\varphi_c\varphi_c'. \tag{6.42}$$

By integrating this equation once with respect to x, we see that the traveling wave (that approaches zero as $x \to \infty$) is given by

$$\varphi_c(x) = \frac{c}{1 + e^{cx}}. \tag{6.43}$$

We note that this traveling wave is a stationary solution of equation (6.41), written in the moving coordinate frame (x, t). To study the stability of the wave, we define $U(y, t) = \varphi(x) + u(x, t)$ and determine the equation of evolution for u:

$$u_t = u_{xx} + c\tanh(\frac{c}{2}x)u_x + \frac{c^2}{2}\text{sech}(\frac{c}{2}x)u - 2uu_x. \tag{6.44}$$

By understanding the stability of the zero solution to this equation, we may determine the stability properties of the traveling wave.

Several remarks should now be made regarding the definition of “stability” for the traveling wave, φ_c. Notice that any translate of the wave, $\varphi_c(x+\gamma)$ for $\gamma \in \mathbb{R}$, is also a solution to equation (6.42). This is due to the fact that Burgers equation is translation invariant, *i.e.* the coefficients of equation (6.41) do not depend on the spatial variable. Thus, we cannot expect that perturbations of the wave will all converge to the same translate. The best we can hope for is referred to as orbital stability, and is defined as follows.

Definition 6.3.1 *The traveling wave* φ_c *is said to be* **asymptotically orbitally stable** *in the Banach space* X *if there exists a* $\delta > 0$ *such that for any initial data satisfying* $||u_0(x) - \varphi_c(x)||_X \leq \delta$ *there exists a* $\gamma \in \mathbb{R}$ *for which*

$$\lim_{t \to \infty} ||u(x, t) - \varphi_c(x + \gamma)||_X = 0.$$

Typically, orbital stability manifests itself in the presence of a one-dimensional set of stationary solutions to the equation of evolution of the perturbation, in this case equation (6.44). These stationary solutions each correspond to a translate of the wave. This one-dimensional set will lead to a one-dimensional center manifold in the phase space of equation (6.44), which will be filled with fixed points. As a result, we cannot expect the zero solution of equation (6.44) to be stable, but we can expect the one-dimensional center manifold to be stable. In other words, if we can show that any solution to equation (6.44) with sufficiently small initial data converges to a point on the one-dimensional center manifold, then the one-dimensional family of translates of the wave will be stable.

Notice that the linear operator in equation (6.44) is the operator that was studied in section 3.4. In the analysis of this operator, it was useful to study its integrated form. We use the same technique here and define

$$v(x, t) = \int_{-\infty}^{x} u(z, t)dz. \tag{6.45}$$

The equation of evolution for v is given by

$$v_t = v_{xx} + c\tanh(\frac{c}{2}x)v_x - (v_x)^2, \tag{6.46}$$

and it is on this equation that we will now focus.

We have not yet specified the Banach space in which we will work. However, most standard choices (*e.g.* $L^2(m)$ or H^1) require that solutions decay to zero at $x = \pm\infty$. Because of the definition of v in equation (6.45), even if $u \in L^2$, for example, v will not necessarily decay as $x \to \infty$. Therefore, analyzing equation (6.46) in L^2 would seem to place some restriction of the allowable functions u. This issue can be resolved through the following observations.

First, notice that for any solution to equation (6.44), $\partial_t \int u(x,t)dx = 0$. Hence, $\int u(x,t)dx = \int u_0(x)dx$. Next, suppose that $\int u_0(x)dx = \int (U(x,0) - \varphi(x))dx = M \neq 0$. If we were to instead study the stability of the translated wave $\varphi_c(x+\delta)$, where $\int (\varphi_c(x+\delta) - \varphi_c(x))\, dx = M$, the new initial data $\tilde{u}_0$ would then satisfy

$$\begin{aligned}
\int \tilde{u}_0(x)dx &= \int (U(x,t) - \varphi_c(x+\delta))\, dx \\
&= \int (U(x,0) - \varphi_c(x))\, dx - \int (\varphi_c(x+\delta) - \varphi_c(x))\, dx \\
&= M - M = 0.
\end{aligned}$$

Therefore, we may assume without loss of generality that $\int u_0(x)dx = 0$.

Effectively what this assumption does is to fix a particular translate of the wave. Thus, we no longer expect to find a one-dimensional center manifold of stationary solutions in the phase space of equation (6.46). The transformation (6.45) has effectively turned orbital stability into "regular" stability. Evidence of this fact can be seen in the spectral pictures of the two linear operators shown in figure 3·2 a) and c). The integrated operator A_2 has no eigenvalue at $\lambda = 0$, while the original operator A_4 does. One can directly check that the eigenvector associated to this zero eigenvalue of A_4 is the derivative of the wave, $\varphi_c'(x) = -\frac{c^2}{4}\text{sech}^2(\frac{c}{2}x)$. It's integral, $\int_{-\infty}^x \varphi_c'(z)dz = \varphi_c(x)$, is not in L^2 and, therefore, not an eigenfunction for A_2.

In order to determine the asymptotic behavior of solutions to equation (6.46), we first state some properties of solutions.

Proposition 6.3.2 *There exists a constant C such that, given any sufficiently small initial data in H^1, the corresponding solution of equation (6.46) satisfies $||v(t)||_{H^1} \leq C||v_0||_{H^1}$ for all $t \geq 0$.*

Proof First, using the theory of fractional Banach spaces [31], one can show that solutions to equation (6.46) exist locally in time in X^α, for $X = L^2$. Because the domain of the linear operator is given by H^2, the embedding theorem 1.6.1 in [31], with $\alpha \in (3/4, 1)$, implies

$$X^\alpha \subset H^s, \text{ for } s < 2\alpha$$
$$X^\alpha \subset C^\nu, \text{ for } 0 \leq \nu < 2\alpha - 1/2. \tag{6.47}$$

Thus, there exists a $T > 0$ such that, for all $0 \leq t < T$, $v(t) \in H^1$. Furthermore, if $T < \infty$, then as $t \to T$, $||v(t)||_\alpha \to \infty$.

We now use the fact that $||v||^2 \equiv \left(\int v^2 + A \int v_x^2\right)$ is equivalent to the H^1 norm. The solution to equation (6.46) satisfies

$$\begin{aligned}
&\frac{d}{dt}\frac{1}{2}\left(\int v^2(x,t)dx + A\int v_x^2(x,t)dx\right) \\
&\quad = -\int v_x^2(x,t) - \frac{c^2}{4}\int \operatorname{sech}^2(\frac{c}{2}x)v^2(x,t)dx - \int v(x,t)v_x^2(x,t)dx \\
&\quad\quad - A\int v_{xx}^2(x,t) + A\frac{c^2}{4}\int \operatorname{sech}^2(\frac{c}{2}x)v_x^2(x,t)dx + A\int v_{xx}(x,t)v_x^2(x,t)dx \\
&\quad \leq -(1 - A\frac{c^2}{4} - ||v(t)||_\infty)\int v_x^2(x,t)dx.
\end{aligned}$$

We remark that the second nonlinear term can be shown to be zero by integrating by parts and noting that the solution has sufficient regularity so that the boundary term vanishes. If we now pick A so that $A < \frac{4}{c^2}$ and T to be the maximal time such that $\sup_{0 \leq t \leq T} ||v(t)|| < 1 - A\frac{c^2}{4}$, then we see that

$$\frac{d}{dt}\frac{1}{2}\left(\int v^2(x,t)dx + A\int v_x^2(x,t)dx\right) \leq 0$$

Hence, if the initial data is such that $||v(0)||_{H^1} \leq 1 - A\frac{c^2}{4}$, the above bound must hold for all $t > 0$. □

We now use the decomposition of solutions that was given in in section 6.2 to determine the asymptotic behavior of solutions to equation (6.46). Define v_1, v_2, and v_3 to be solutions of

$$\partial_t v_1 = \partial_x^2 v_1 + c\partial_x v_1 - \frac{[(1 + e^{-cx})\partial_x v_1 + ce^{-cx}v_1]^2}{(1 + e^{-cx})^3}, \tag{6.48}$$

$$\partial_t v_2 = \partial_x^2 v_2 - c\partial_x v_2 - \frac{[(1 + e^{+cx})\partial_x v_2 - ce^{+cx}v_2]^2}{(1 + e^{+cx})^3}, \tag{6.49}$$

$$\partial_t v_3 = \mathcal{L}_3 v_3 - N_3(v_3) - F_3(x,t), \tag{6.50}$$

with initial data given by

$$v_{1,2}(x,0) = v(x,0) - \operatorname{sech}(\frac{c}{2}x)v(x,0)$$
$$v_3(x,0) = \operatorname{sech}(\frac{c}{2}x)v(x,0). \qquad (6.51)$$

In the above equations,

$$\begin{aligned}
\mathcal{L}_3 v_3 &= \partial_x^2 v_3 + \left[c\tanh(\frac{c}{2}x) - 2\left(a(x,t) + b(x,t)\right)\right]\partial_x v_3 \\
N_3(v_3) &= (\partial_x v_3)^2 \\
F_3(x,t) &= 2a(x,t)b(x,t) \qquad (6.52)\\
a(x,t) &= \frac{\partial_x v_1(x,t)}{(1+e^{-cx})} + \frac{ce^{-cx}}{(1+e^{-cx})^2}v_1(x,t) \\
b(x,t) &= \frac{\partial_x v_2(x,t)}{(1+e^{+cx})} - \frac{ce^{+cx}}{(1+e^{+cx})^2}v_2(x,t).
\end{aligned}$$

Notice that

$$v(x,t) = \frac{1}{(1+e^{-cx})}v_1(x,t) + \frac{1}{(1+e^{+cx})}v_2(x,t) + v_3(x,t) \qquad (6.53)$$

is the solution to equation (6.46). (See remark 6.2.1.) We will determine the asymptotic (in time) behavior of v by determining that of v_1, v_2, and v_3. As in section 6.2, we first study the evolution of v_1 and v_2 using scaling variables. We state the results for v_1. Those for v_2 are similar.

Define the scaling transformation

$$v_1(x,t) = \frac{1}{(t+1)}w_1(\frac{x+c(t+1)}{\sqrt{t+1}}, \log(t+1))$$
$$\eta = \frac{x+c(t+1)}{\sqrt{t+1}}, \quad \tau = \log(t+1). \qquad (6.54)$$

The equation of evolution for w_1 is given by

$$\partial_\tau w_1 = \mathcal{L}w_1 + \frac{1}{2}w_1 - N(\eta,\tau,w_1), \qquad (6.55)$$

where $\mathcal{L}w_1 = \partial_\eta^2 w_1 + \frac{1}{2}\eta\partial_\eta w_1 + \frac{1}{2}w_1$ and

$$\begin{aligned} N(\eta,\tau,w_1) = & \frac{e^{-\tau}}{(1+e^{-c\gamma(\eta,\tau)})}(\partial_\eta w_1)^2 + \frac{2ce^{-\frac{1}{2}\tau}e^{-c\gamma(\eta,\tau)}}{(1+e^{-c\gamma(\eta,\tau)})^2}w_1\partial_\eta w_1 \\ & + \frac{c^2 e^{-2c\gamma(\eta,\tau)}}{(1+e^{-c\gamma(\eta,\tau)})^3}w_1^2, \end{aligned} \tag{6.56}$$

with $\gamma(\eta,\tau) = \eta e^{\frac{\tau}{2}} - ce^{\tau}$.

In order to apply the center manifold theorem to equation (6.55), we will use the variable ζ defined via equation (6.31) in section 6.2. We may then write equation (6.55) as

$$\begin{aligned} \partial_\tau w_1 &= \mathcal{L}w_1 + \frac{1}{2}w_1 - N(w_1,\eta,\zeta) \\ \partial_\tau \zeta &= = -\zeta + \frac{1}{2}\zeta^2. \end{aligned} \tag{6.57}$$

Notice that $\zeta = 0$ is stable, and so this additional equation will simply contribute to the stable manifold. Furthermore, notice that, in the equation for w_1, the largest eigenvalue is now $\lambda_0 = 1/2$, due to the additional linear term. Thus, to obtain the leading order asymptotics, we would like to apply the center manifold theorem, theorem 5.2.1, to equation (6.57) with $\sigma_c = \{1/2\}$ and $\sigma_s = \sigma(\mathcal{L}) \setminus \sigma_c$.

To do so, we must show that the hypothesis 1) - 4) of theorem 5.2.1 are satisfied. Due to the form of the nonlinearity in (6.57), we will need to work in $X = H^1(m)$, where $H^1(m) = \{u : u, \partial_\eta u \in L^2(m)\}$, rather than $L^2(m)$. Although this requires more regularity in the initial data, this is somewhat natural, as we are working with the integrated form of equation (6.44).

Proposition 6.3.3 *Fix $T > 0$ and m. For any $w_1 \in C^0([0,T], H^1(m))$, define*

$$R(\tau) = \int_0^\tau e^{\mathcal{L}(\tau-s)}N(w_1(s))ds. \tag{6.58}$$

Then $R(\tau) \in C^0([0,T], H^1(m))$, and there exists a $C(m, r_0, T)$ such that, if w_1, $\tilde{w}_1 \in C^0([0,T], H^1(m))$ with $\sup_{0\le\tau\le T}||w_1(\tau)||_{H^1(m)} \le r_0$ and $\sup_{0\le\tau\le T}||\tilde{w}_1(\tau)||_{H^1(m)} \le r_0$, then the corresponding integral terms satisfy

$$\sup_{0\le\tau\le T}||R(\tau) - \tilde{R}(\tau)||_{H^1(m)} \le C(m,T,r_0)\sup_{0\le\tau\le T}||w_1(\tau) - \tilde{w}_1(\tau)||_{H^1(m)}.$$

Furthermore, the constant $C(m,T,r_0) \to 0$ as $T \to 0$ and as $r_0 \to 0$.

Proof The proof of this proposition is similar to that of proposition 6.1.3. Note that the nonlinearity in the w_1 component of equation (6.57) may be written

$$N(w_1,\eta,\zeta) = b_1(\eta,\zeta)(\partial_\eta w_1)^2 + b_2(\eta,\zeta)w_1\partial_\eta w_1 + b_3(\eta,\zeta)w_1^2,$$

where $b_i(\eta,\zeta)$, $i = 1,2,3$, are smooth, bounded functions. Using the bound on the semi-group given in equation (6.12), we have that

$$\begin{aligned}
&\int_0^\tau ||e^{\mathcal{L}(\tau-s)}(N(w_1(s)) - N(\tilde{w}_1(s)))||_{L^2(m)}ds \\
&\quad\le \int_0^\tau \frac{C}{(a(\tau-s))^{\frac{1}{2}(1-\frac{1}{2})}}||(\partial_\eta w_1(s))^2 - (\partial_\eta \tilde{w}_1(s))^2||_{L^1(m)}ds \\
&\qquad + \int_0^\tau \frac{C}{(a(\tau-s))^{\frac{1}{2}(1-\frac{1}{2})}}||w_1(s)\partial_\eta w_1(s) - \tilde{w}_1(s)\partial_\eta \tilde{w}_1(s)||_{L^1(m)}ds \\
&\qquad + \int_0^\tau \frac{C}{(a(\tau-s))^{\frac{1}{2}(1-\frac{1}{2})}}||w_1^2(s) - \tilde{w}_1^2(s)||_{L^1(m)}ds \\
&\quad\le \int_0^\tau \frac{C}{(a(\tau-s))^{\frac{1}{2}(1-\frac{1}{2})}}\left(||w_1(s)||_{H^1(m)} + ||\tilde{w}_1(s)||_{H^1(m)}\right)||w_1(s) - \tilde{w}_1(s)||_{H^1(m)}ds \\
&\quad\le \left(\sup_{0\le\tau\le T}\int_0^\tau \frac{C}{(a(\tau-s))^{\frac{1}{2}(1-\frac{1}{2})}}ds\right)\left(\sup_{0\le\tau\le T}||w_1(\tau)||_{H^1(m)} + \sup_{0\le\tau\le T}||\tilde{w}_1(\tau)||_{H^1(m)}\right) \\
&\qquad \times \left(\sup_{0\le\tau\le T}||w_1(\tau) - \tilde{w}_1(\tau)||_{H^1(m)}\right).
\end{aligned}$$

The term

$$\int_0^\tau ||\partial_\eta e^{\mathcal{L}(\tau-s)}(N(w_1(s)) - N(\tilde{w}_1(s)))||_{L^2(m)}ds$$

may be bounded similarly, using the estimate in equation (6.12) with $\alpha = 1$. □

Write equation (6.57) as

$$\partial_t W = \Lambda W + \mathcal{N}(W,\eta), \tag{6.59}$$

where

$$W = \begin{pmatrix} w_1 \\ \zeta \end{pmatrix},\ \Lambda = \begin{pmatrix} \mathcal{L} & 0 \\ 0 & -1 \end{pmatrix},\ \mathcal{N}(W,\eta) = \begin{pmatrix} \frac{1}{2}w_1 - N(w_1,\eta,\zeta) \\ \frac{1}{2}\zeta^2 \end{pmatrix}. \tag{6.60}$$

As in sections 6.1 and 6.2 we will need to cutoff the nonlinearity outside of a neighborhood of the origin in $H^1(m)$. Let $\chi_{r_0}(w_1) : H^1(m) \to \mathbb{R}^+$ be a smooth function satisfying $\chi_{r_0}(w_1) = 1$ if $||w_1||_{H^1(m)} \leq r_0$, and $\chi_{r_0}(w_1) = 0$ if $||w_1||_{H^1(m)} \geq 2r_0$. We will construct a center-unstable manifold for the following equation:

$$\partial_t W = \Lambda W + \mathcal{N}_{r_0}(W, \eta), \tag{6.61}$$

where

$$\mathcal{N}_{r_0}(W, \eta) = \begin{pmatrix} \frac{1}{2}w_1 - \chi_{r_0}(w_1) N(w_1, \eta, \zeta) \\ \frac{1}{2}\zeta^2 \end{pmatrix}. \tag{6.62}$$

Proposition 6.3.4 *Given any sufficiently small r_0 and sufficiently small $w_0 \in H^1(m)$, there exists a solution to equation (6.61) satisfying $w(\tau) \in C^0([0, \infty), H^1(m))$.*

Proof Due to the instability of the spectrum, *i.e.* the eigenvalue at $\lambda = 1/2$, we cannot expect the solutions to the equation to be bounded for all time. However, they will turn out to exist for all time, meaning that their $H^1(m)$ norm does not become unbounded in finite time. To see this, we utilize the fact that the nonlinearity has been cutoff outside of a neighborhood of the origin. Local existence for the nonlinear equation can be obtained using proposition 6.3.3 and a fixed point argument. On the other hand, solutions to the linear equation exist globally in time. Therefore, if a solution to the nonlinear equation was to leave the ball of radius $2r_0$ in $H^1(m)$, then the nonlinearity will become zero, leaving only the linear evolution. As a result, solutions cannot become unbounded in finite time. $\square$

We can now use propositions 6.3.4 and 6.3.3 to show that the hypotheses to theorem 5.2.1 are satisfied. In particular, we have the following result.

Proposition 6.3.5 *Let $\Phi_1^{r_0}$ be the semiflow associated to equation (6.61) at time $\tau = 1$. Then, if $r_0 > 0$ is sufficiently small, the semigroup can be decomposed as*

$$\Phi_1^{r_0} = L + \mathcal{R},$$

where L is a bounded linear map, and $\mathcal{R}$ is a globally Lipschitz map such that Lip($\mathcal{R}$) $\leq C(r_0)$ where $C(r_0) \to 0$ as $r_0 \to 0$. Furthermore, $\mathcal{R}$ is C^1 with $\mathcal{R}(0) = D\mathcal{R}(0) = 0$.

Hence, we know that there exists a one-dimensional center-unstable manifold in the phase space of equation (6.61). Any solution on this manifold can be written $(w_1(\eta, \tau), \zeta(\tau)) = \alpha(\tau)\phi_0(\eta) + g_1(\alpha)$, where ϕ_0 is the eigenfunction of $\mathcal{L}$ associated to the zero eigenvalue, given in proposition 3.1.4, and $g_1(\alpha) = \mathcal{O}(\alpha^2)$. Therefore, we find that the dynamics on the center manifold are given by

$$\dot{\alpha} = \frac{1}{2}\alpha - P_c\left[\chi_{r_0} N(\alpha\phi_0 + g_1(\alpha), \eta, \zeta)\right], \tag{6.63}$$

where P_c is the projection operator associated to the zero eigenvalue of $\mathcal{L}$, given in equation (6.19). Due to the explicit dependence of the nonlinearity, N, on both η and ζ, it is difficult to compute the coefficient of the nonlinearity in equation (6.63) explicitly. However, because the coefficients of N are smooth and bounded, one can show that equation (6.63) must have the form

$$\dot{\alpha} = \frac{1}{2}\alpha - B_1\alpha^2 + \mathcal{O}(\alpha^3), \tag{6.64}$$

for some constant $B_1 > 0$. We assume that B_1 is sufficiently large so that the fixed point $\frac{1}{2B_1}$ lies within the domain in which the center manifold was constructed. Therefore, $\alpha(\tau) = \frac{1}{2B_1} + \mathcal{O}(e^{-\frac{\tau}{2}})$, and $w_1(\eta,\tau) = \frac{1}{2B_1}\phi_0 + g_1(\frac{1}{2B_1}) + \mathcal{O}(e^{-\frac{\tau}{2}})$. Transforming back to the original (x,t) variables, we find that the asymptotic form of v_1 is given by

$$v_1(x,t) = \frac{1}{2B_1\sqrt{4\pi}(t+1)}e^{-\frac{(x+c(t+1))^2}{4(t+1)}} + \frac{g_1(\frac{1}{2B_1})}{t+1} + \mathcal{O}((t+1)^{-\frac{3}{2}}). \tag{6.65}$$

Note that the presence of the term $\frac{g_1(\frac{1}{2B_1})}{t+1}$ implies that solutions have to a nontrivial, self-similar form that is, in general, not strictly a Gaussian. However, we will assume that the form of $g_1(1/2B_1)$ is Gaussian-like. This assumption is motivated by the fact that the self-similar solutions to equation (6.6) are Gaussian-like for $1 < p < 3$, and the most relevant component of the nonlinearity in equation (6.55) is given by w_1^2. Similarly, one can show that

$$v_2(x,t) = \frac{1}{2B_2\sqrt{4\pi}(t+1)}e^{-\frac{(x-c(t+1))^2}{4(t+1)}} + \frac{g_2(\frac{1}{2B_2})}{t+1} + \mathcal{O}((t+1)^{-\frac{3}{2}}), \tag{6.66}$$

where $g_2(1/2B_2)$ is again Gaussian-like.

This calculation suggests that the relevance of nonlinearities containing derivatives is different for the linear operator in equation (6.44) than for that of equation (6.1). In particular, due to the structure of the decomposition in equation (6.53), when a derivative gets applied to v, terms result in which the derivative gets applied only to the exponential prefactors, rather than the component functions themselves. Therefore, in some sense the derivative is no less relevant than the function itself, when it appears in a polynomial nonlinearity.

On the other hand, the Gaussian and Gaussian-like terms in the expansions for v_1 and v_2 actually decay exponentially in time in the original (x,t) variables. When combined with the exponential prefactors in the decomposition of solutions given in equation (6.53), we see that

$$\frac{e^{-\frac{(x+c(t+1))^2}{4(t+1)}}}{1+e^{-cx}} = \operatorname{sech}(\frac{c}{2}x)e^{-\frac{c^2}{4}(t+1)}e^{-\frac{x^2}{4(t+1)}}. \tag{6.67}$$

This implies that the temporal decay of $\frac{v_1}{1+e^{-cx}}$ and $\frac{v_2}{1+e^{-cx}}$ will actually be governed by the seemingly higher order terms in equations (6.65) and (6.65). If the initial data lies in $L^2(m)$, we have that

$$\frac{v_1(x,t)}{1+e^{-cx}} + \frac{v_2(x,t)}{1+e^{+cx}} = \mathcal{O}((t+1)^{-\frac{2m+1}{4}}). \tag{6.68}$$

As a result, the nonlinearity in equation (6.46) does not play a key role in determining the asymptotic dynamics of solutions. Hence, one could argue that it is, in fact, irrelevant.

We remark that an intuitive explanation for this behavior is as follows. Recall that in section 3.2 the relationship between the inflowing nature of the linear operator $\partial_x^2 + \tanh(\frac{c}{2}x)\partial_x$ and the decay rate of solutions was discussed. It is the asymptotic (in space) structure of the linear operator that determines if it is inflowing or outflowing. In turn, the asymptotic structure of the operator is directly related to its essential spectrum. Therefore, it makes sense that it should be the location of essential spectrum that governs the large time behavior of solutions.

We must now determine the dynamics of v_3. We will analyze this equation using an exponential weight, as in section 6.2. This technique was introduced in [52] to push the essential spectrum of the linear operator off the imaginary axis, which allows for more control of the behavior of solutions. Note that this will not impose any exponential decay requirements on the initial data for v. (See remark 6.2.1.) Define $w_3 \equiv \cosh(\frac{c}{2}x)v_3$. The equation of evolution for w_3 is given by

$$\partial_t w_3 = \tilde{\mathcal{L}}_3 w_3 - \tilde{N}_3(w_3, x) - \tilde{F}_3(x,t), \tag{6.69}$$

where

$$\begin{aligned}
\tilde{\mathcal{L}}_3 w_3 &= \partial_x^2 w_3 - 2(a(x,t)+b(x,t))\partial_x w_3 - \left(\frac{c^2}{4} - c\tanh(\frac{c}{2}x)(a(x,t)+b(x,t))\right) w_3 \\
\tilde{N}_3(w_3,x) &= \operatorname{sech}(\frac{c}{2}x)\left(-\frac{c}{2}\tanh(\frac{c}{2}x)w_3 + \partial_x w_3\right)^2 \\
\tilde{F}_3(x,t) &= \cosh(\frac{c}{2}x)F_3(x,t).
\end{aligned} \tag{6.70}$$

In order to analyze the evolution of w_3, we will split up the linear operator as

$$\begin{aligned}\tilde{\mathcal{L}}_3 w_3 &= (\partial_x^2 - \frac{c^2}{4})w + B(t)w \\ B(t)w_3 &= -2(a(x,t)+b(x,t))\partial_x w_3 + c\tanh(\frac{c}{2}x)(a(x,t)+b(x,t))w_3 \end{aligned} \tag{6.71}$$

and study the following integral form of the solution:

$$\begin{aligned} w_3(t) = e^{(\partial_x^2 - \frac{c^2}{4})t} w_3(0) &- \int_0^t e^{(\partial_x^2-\frac{c^2}{4})(t-s)} B(s) w_3(s) ds - \int_0^t e^{(\partial_x^2-\frac{c^2}{4})(t-s)} \tilde{N}_3(w_3(s)) ds \\ &- \int_0^t e^{(\partial_x^2-\frac{c^2}{4})(t-s)} \tilde{F}_3(s) ds. \end{aligned} \tag{6.72}$$

Intuitively, one can see that the large time evolution of w_3 should be governed by the last term on the right hand side of the above equation. This is because the semigroup $e^{(\partial_x^2-\frac{c^2}{4})t}$ decays exponentially in time, and so the first three terms on the right hand side of equation (6.72) will, as well. Therefore, the behavior of solutions will be governed by the inhomogeneity, $\tilde{F}_3$.

To see this rigorously, we first must prove some existence results for equation (6.69).

Proposition 6.3.6 *Given any sufficiently small initial data in H^1, there exists a $T > 0$ and a solution to equation (6.72) satisfying $w_3(t) \in C^0([0,T), H^1)$.*

Proof Consider the Banach space $Y = \{w(t) \in C^0([0,T), H^1)$ such that $w(0) = w_3(0)\}$, equipped with the norm $||w||_Y = \sup_{0\le t\le T} ||w(t)||_{H^1}$, and let $Y_r(w_3(0))$ be the ball of radius r in Y centered at $w_3(0)$. Define the map

$$\begin{aligned} G(w)(t) = e^{(\partial_x^2 - \frac{c^2}{4})t} w_3(0) &- \int_0^t e^{(\partial_x^2-\frac{c^2}{4})(t-s)} B(s) w(s) ds - \int_0^t e^{(\partial_x^2-\frac{c^2}{4})(t-s)} \tilde{N}_3(w(s)) ds \\ &- \int_0^t e^{(\partial_x^2-\frac{c^2}{4})(t-s)} \tilde{F}_3(s) ds. \end{aligned}$$

Local existence of solutions will follow if we can show that $G : Y_r \to Y_r$ and is a contraction there. To see this, first note that

$$||e^{(\partial_x^2-\frac{c^2}{4})t} u||_{H^1} \le C e^{-\frac{c^2}{4}t} \max\left(\frac{1}{t^{\frac{1}{4}}}, \frac{1}{t^{\frac{3}{4}}}\right) ||u||_{L^1}.$$

Next, we may bound

$$\begin{aligned} ||B(s)(w(s) - \tilde{w}(s)||_{L^1} &\le C(||v_1(s)||_{H^1} + ||v_2(s)||_{H^1})||w(s) - \tilde{w}(s)||_{H^1} \\ ||\tilde{N}_3(w(s)) - \tilde{N}_3(\tilde{w}(s))||_{L^1} &\le C(||w(s)||_{H^1} + ||\tilde{w}(s)||_{H^1})||w(s) - \tilde{w}(s)||_{H^1} \\ ||\tilde{F}_3(s)||_{L^1} &\le C(||v_1(s)||_{H^1} + ||v_2(s)||_{H^1}). \end{aligned} \tag{6.73}$$

In addition, from the expansions of v_1 and v_2 given in equations (6.65) and (6.66), we have that

$$||v_i(t)||_{H^1} \le C. \tag{6.74}$$

We now show that $G : Y_r \to Y_r$ and is continuous.

$$\begin{aligned} ||G(w(t)) - w_3(0)||_{H^1} &\le ||e^{(\partial_x^2 - \frac{c^2}{4})t} w(t) - w_3(0)||_{H^1} \\ &+ \int_0^t \frac{Ce^{-\frac{c^2}{4}t}}{(t-s)^{\frac{3}{4}}} (||v_1(s)||_{H^1} + ||v_2(s)||_{H^1})||w(s)||_{H^1} ds + \int_0^t \frac{Ce^{-\frac{c^2}{4}t}}{(t-s)^{\frac{3}{4}}} ||w(s)||^2_{H^1} ds \\ &+ \int_0^t \frac{Ce^{-\frac{c^2}{4}t}}{(t-s)^{\frac{3}{4}}} (||v_1(s)||_{H^1} + ||v_2(s)||_{H^1}) ds. \end{aligned}$$

To bound the first term, recall that $e^{(\partial_x^2 - \frac{c^2}{4})t}$ generates an analytic semigroup on L^2, and is therefore pointwise continuous there. As a result, we may bound

$$\begin{aligned} ||e^{(\partial_x^2 - \frac{c^2}{4})t} w(t) - w_3(0)||^2_{H^1} &\le ||e^{(\partial_x^2 - \frac{c^2}{4})t} w(t) - w_3(0)||^2_{L^2} \\ &\quad + ||e^{(\partial_x^2 - \frac{c^2}{4})t} \partial_x w(t) - \partial_x w_3(0)||^2_{L^2} \\ &\le C(T) \end{aligned}$$

where $C(T) \to 0$ as $T \to 0$.

To bound the remaining three terms, use equation (6.74) and the fact that

$$\int_0^t \frac{Ce^{-\frac{c^2}{4}t}}{(t-s)^{\frac{3}{4}}} ds \le CT^{\frac{1}{4}}$$

for all $0 \le t \le T$.

To see that G is a contraction, we compute

$$||G(w)(t) - G(\tilde{w}(t))||_{H^1} \leq \int_0^t \frac{Ce^{-\frac{c^2}{4}t}}{(t-s)^{\frac{3}{4}}} C(||v_1(s)||_{H^1} + ||v_2(s)||_{H^1})||w(s) - \tilde{w}(s)||_{H^1}$$
$$+ \int_0^t \frac{Ce^{-\frac{c^2}{4}t}}{(t-s)^{\frac{3}{4}}} C(||w(s)||_{H^1} + ||\tilde{w}(s)||_{H^1})||w(s) - \tilde{w}(s)||_{H^1}.$$

Hence,

$$\sup_{0\leq t\leq T} ||G(w)(t) - G(\tilde{w}(t))||_{H^1} \leq C(T, r_0) \sup_{0\leq t\leq T} ||w(s) - \tilde{w}(s)||_{H^1},$$

which proves the result. □

We now show that the inhomogeneity governs the large time behavior of w_3. Recall the asymptotic expansions of v_1 and v_2, given in equations (6.65) and (6.66). The Gaussian-like leading order terms actually decay exponentially in the (x, t) variables, and, as a result, the temporal decay rate of the solution will be governed by the seemingly higher order terms. If one works in the space $H^1(m)$, these terms decay at a rate given by $e^{-\frac{2m+1}{4}\tau}$. Therefore, we have

$$||v_{1,2}(t)||_{H^1} \leq \frac{C(v_{1,2}(0))}{(t+1)^{\frac{2m+1}{4}}}, \tag{6.75}$$

where $C(v_{1,2}(0)) \to 0$ as $||v_{1,2}(0)||_{H^1(m)} \to 0$. Note that this estimate is in terms of the H^1 norm because $||v_{1,2}(t)||_{H^1} \leq C||w_{1,2}(\tau)||_{H^1(m)}$, and the asymptotic expansions for $w_{1,2}$ were carried out in the space $H^1(m)$. The main idea is to show that $||w_3(t)|| \leq C(t+1)^{-\frac{2m+1}{4}}$ for $t \geq 0$, as well.

Proposition 6.3.7 *Let $v_{1,2} \in C^0([0,\infty), H^1(m))$ be the solutions to equations (6.48) and (6.49) with initial data $v_{1,2}(0) \in H^1(m)$. Let $w_3 \in C^0([0,T), H^1)$ be the solution to equation (6.69) with initial data $w_3(0) \in H^1$. Then $w_3 \in C^0([0,\infty), H^1)$ and satisfies*

$$||w_3(t)||_{H^1} \leq \frac{C}{(t+1)^{\frac{2m+1}{4}}}.$$

Proof Using the integral form of the solution given in equation (6.72), the fact that $||e^{(\partial_x^2 - \frac{c^2}{4})t}u||_{H^1} \leq Ce^{-\frac{c^2}{4}t}||u||_{H^1}$, and the bounds given in equation (6.73), we may estimate

$$\begin{aligned}||w_3(t)||_{H^1} \le Ce^{-\frac{c^2}{4}t}||w_3(0)||_{H^1} &+ C\int_0^t \frac{e^{-\frac{c^2}{4}(t-s)}}{(t-s)^\beta}(||v_1(s)||_{H^1}+||v_2(s)||_{H^1})||w(s)||_{H^1}ds \\ &+ C\int_0^t \frac{e^{-\frac{c^2}{4}(t-s)}}{(t-s)^\beta}||w(s)||^2_{H^1}ds \\ &+ C\int_0^t \frac{e^{-\frac{c^2}{4}(t-s)}}{(t-s)^\beta}(||v_1(s)||_{H^1}+||v_2(s)||_{H^1})ds,\end{aligned}$$

where $0<\beta<1$. Define $|||w_3||| = \sup_{0\le t\le T}(t+1)^{\frac{2m+1}{4}}||w_3(t)||_{H^1}$. We then have

$$\begin{aligned}|||w_3||| \le C \sup_{0\le t\le T}&(t+1)^{\frac{2m+1}{4}}e^{-\frac{c^2}{4}t}||w_3(0)||_{H^1} \\ &+ C(v_{1,2}(0)) \sup_{0\le t\le T}(t+1)^{\frac{2m+1}{4}}\int_0^t \frac{e^{-\frac{c^2}{4}(t-s)}}{(t-s)^\beta}\frac{1}{(s+1)^{\frac{2m+1}{2}}}ds|||w_3||| \\ &+ C \sup_{0\le t\le T}(t+1)^{\frac{2m+1}{4}}\int_0^t \frac{e^{-\frac{c^2}{4}(t-s)}}{(t-s)^\beta}\frac{1}{(s+1)^{\frac{2m+1}{2}}}ds|||w_3|||^2 \\ &+ C(v_{1,2}(0)) \sup_{0\le t\le T}(t+1)^{\frac{2m+1}{4}}\int_0^t \frac{e^{-\frac{c^2}{4}(t-s)}}{(t-s)^\beta}\frac{1}{(s+1)^{\frac{2m+1}{4}}}ds.\end{aligned}$$

Therefore, we see that

$$|||w_3|||\,(1 - C(v_{1,2}(0)) - M_1|||w_3|||) \le M_2||w_3(0)||_{H^1} + C(v_{1,2}(0)).$$

If $v_{1,2}(0)$ are chosen sufficiently small in $H^1(m)$ so that $1/2 - C(v_{1,2}(0)) > 0$ and $C(v_{1,2}(0)) \le [1/2 - C(v_{1,2}(0))]/(4M_1)$, and T is chosen to be the maximal time such that $|||w_3||| \le [1/2 - C(v_{1,2}(0))]/M_1$, then we have that

$$|||w_3||| \le 2M_2||w_3(0)||_{H^1} + 2C(v_{1,2}(0)).$$

Therefore, if $||w_3(0)||_{H^1} \le (1/2 - C(v_{1,2}(0)))/(4M_1M_2)$, then the bound must hold for all $t \ge 0$. □

We may now combine the asymptotic results for v_1, v_2, and v_3, to obtain

Theorem 6.3.8 *Suppose that $v(x,0) \in H^1(m)$ is sufficiently small. Then the correspond-*

ing solution to equation (6.46) satisfies $v \in C^0([0,\infty), H^1)$ *with*

$$||v(t)||_{H^1} \leq \frac{C}{(t+1)^{\frac{2m+1}{4}}}. \tag{6.76}$$

We note that this result and the method used to obtain it are related to the results of [35] and [60], respectively. In [35] it was shown the the traveling wave solution to Burgers equation is stable in algebraically weighted L^∞ spaces, with an algebraic decay rate similar to the one in equation (6.76). The main advantage to using the decomposition of the solution, outlined above, is that it provides additional information on the underlying structure that governs the decay of perturbations.

Results of a slightly different type were given in [60]. In this paper, the authors provided an explicit condition for the nonlinear stability of the wave, which was obtained using detailed bounds on the Green's function associated to the resolvent operator. These bounds were proved by decomposing the Green's functions using its scattering structure. One similarity between the approach considered here and that in [60] is that in both cases a more detailed view of the asymptotics is obtained by decomposing the evolution of the perturbation into "near-filed" and "far-field" components.

Chapter 7

Global Stability

In this chapter the global stability of solutions to PDEs is discussed. First, we present an introduction to Lyapunov functionals and LaSalle's Invariance Principle. Next, those ideas are used to demonstrate that the traveling front in Burgers equation is globally stable.

7.1 Lyapunov Functionals and LaSalle's Invariance Principle

In this section a brief introduction to Lyapunov functionals and LaSalle's Invariance Principle is given. The presentation follows that of [31]. Although the results are stated for Banach spaces, we remark that one could work in any complete metric space. Throughout this discussion, the semiflow generated by a nonlinear PDE will be referred to as a dynamical system and be denoted by $\{S(t)\}_{t\geq 0}$. In other words, if $u(x,t)$ is the solution to a nonlinear PDE with initial data u_0, then we denote it by $u(t) = S(t)u_0$. We use the notion $S(t)$, rather than $T(t)$, to distinguish it from the flow generated by the linear semigroup. We begin with some definitions.

Definition 7.1.1 *Let $\{S(t)\}_{t\geq 0}$ be a dynamical system on a Banach space X. For any $u \in X$, the* **orbit** *through u is given by*

$$\gamma(u) = \{S(t)u, \quad t \geq 0\}.$$

We say that u is an **equilibrium point** *(or stationary solution) if $\gamma(u) = u$. It is* **stable** *if for any $\epsilon > 0$ there exists a $\delta = \delta(\epsilon) > 0$ such that for all $t \geq 0$*

$$||u - v||_X < \delta \Rightarrow ||S(t)v - u||_X < \epsilon.$$

It is **asymptotically stable** *if δ can be chosen so that $||S(t)v - u||_X \to 0$ as $t \to \infty$.*

Next, we define a Lyapunov functional.

Definition 7.1.2 *A* **Lyapunov functional** *is a continuous function $V : X \to \mathbb{R}$ such that*

$$\dot{V}(u) = \limsup_{t\to 0^+} \frac{1}{t}\left(V(S(t)u) - V(u)\right) \leq 0,$$

for all $u \in X$.

The standard theorem relating Lyapunov functionals to the stability of an equilibrium point is as follows.

Theorem 7.1.3 *[31] Suppose $u \equiv 0$ is an equilibrium solution of a dynamical system $\{S(t)\}_{t\geq 0}$ on X. In addition, suppose there exists a continuous, strictly increasing function $c_1 : \mathbb{R} \to \mathbb{R}$ satisfying $c_1(0) = 0$ and $c_1(r) > 0$ if $r > 0$ such that $V(0) = 0$ and $V(u) \geq c_1(||u||_X)$. Then $u \equiv 0$ is stable. Furthermore, if $\dot{V}(u) \leq -c_2(||u||)$, where c_2 is also continuous, increasing, and positive with $c_2(0) = 0$, then $u \equiv 0$ is asymptotically stable.*

Although it is not explicitly stated, this theorem implies that the basin of attraction of $u \equiv 0$ includes any subset of X for which the assumptions hold. Hence, if the assumptions hold for all $u \in X$, then 0 is globally stable.

Although this result is quite nice, it is typically difficult to find a Lyapunov functional satisfying the additional assumptions stated in the above theorem. As a result, we often work instead with a result known as LaSalle's Invariance Principle, as it has less stringent assumptions. In order to state the theorem, we'll need the following definitions.

Definition 7.1.4 *A set $K \subset X$ is invariant with respect to the dynamical system $\{S(t)\}_{t\geq 0}$ if for any $u_0 \in K$ there exists a continuous curve $u : \mathbb{R} \to K$ such that $u(0) = u_0$ and*

$$S(t)u(\tau) = u(t+\tau), \text{ for } \tau \in \mathbb{R} \text{ and } t \geq 0.$$

Definition 7.1.5 *The* **omega limit set**, *or ω-limit set, of u is defined to be*

$$\omega(u) = \{v \in X : \exists t_n \to \infty \text{ such that } S(t_n)u \to v\}.$$

Theorem 7.1.6 LaSalle's Invariance Principle *[31] Suppose V is a Lyapunov functional on X. Let $E = \{u \in X : \dot{V}(u) = 0\}$ and M be the maximal invariant subset of E. If u_0 is any initial data whose orbit is contained in a compact subset of X, then $S(t)u_0 \to M$ as $t \to \infty$.*

Therefore, if one can prove that all solutions to a given nonlinear PDE with initial data in X have orbits that are contained in a compact subset of X, then the set M is globally stable.

As mentioned above in section 6.3, when studying the stability of traveling waves one is often interested in the orbital stability of the wave. Thus, stability of the wave means that solutions converges to some translate of the wave. This family of translates of the wave manifests itself as a center manifold of stationary solutions in the phase space of perturbations of the wave. Thus, perturbations typically approach some member of this set of stationary solutions, which, in the context of Burgers equation, will be represented by the set M in the above theorem.

The compactness assumption is necessary to insure that trajectories have a limit point and don't "run off to infinity." Roughly speaking, this is due to the fact that any continuous, decreasing function that is defined on a compact set must have a limit point.

7.2 Global Stability of the Traveling Front in Burgers Equation

We are interested in studying the global stability of traveling fronts in Burgers Equation. Local stability of these solutions was studied in section 6.3, and for convenience we restate here those results that will be relevant for the global stability argument.

Burgers equation is given by

$$\partial_t U(y,t) = \partial_y^2 U(y,t) - \partial_y (U^2(y,t)), \tag{7.1}$$

and the traveling front, which is a stationary solution with respect to the moving coordinate $x = y - ct$, is given by

$$\varphi_c(x) = \frac{c}{1+e^{cx}}. \tag{7.2}$$

Recall that, in order to study stability, we define $U(y,t) = \varphi(x) + u(x,t)$ and study the evolution of u, which is given by the equation

$$u_t = u_{xx} + c\tanh(\frac{c}{2}x)u_x + \frac{c^2}{2}\text{sech}^2(\frac{c}{2}x)u - 2uu_x. \tag{7.3}$$

7.2.1 Definition of the Lyapunov Functional

One tool that can be used in studying Burgers equation is the Cole-Hopf transformation, which turns the nonlinear Burgers equation into the linear heat equation. Because equation (7.3) is related to Burgers equation, one might think that applying the Cole-Hopf transformation to it would be useful, as well. To do so, define

$$w(x,t) = u(x,t)\text{exp}[-\int_{-\infty}^{x} u(z,t)dz], \tag{7.4}$$

and apply this transformation to (7.3). We find that

$$w_t = w_{xx} + c\tanh(\frac{c}{2}x)w_x + \frac{c^2}{2}\text{sech}^2(\frac{c}{2}x)w. \tag{7.5}$$

Note that the Cole-Hopf transformation has simply removed the nonlinear term from equation (7.3). This is extremely useful because equation (7.5) is a Fokker-Plank equation, *i.e.* an equation of the form

$$w_t = \partial_x \left(w_x + W'(x)w\right), \tag{7.6}$$

where the function $W(x)$ is known as the potential function. For equation (7.5), we have that $W(x) = 2\log\left[\cosh(\frac{c}{2}x)\right]$.

It is known that for equation (7.6) there is a one parameter family of stationary solutions, $w_\infty(x) = \alpha e^{-W(x)}$, for $\alpha \in \mathbb{R}$. These solutions can be shown to be globally stable by considering the Lyapunov functional defined by

$$H[w](t) = \int_{-\infty}^{+\infty} w(x,t)\log[\frac{w(x,t)}{e^{-W(x)}}]dx. \tag{7.7}$$

For global stability arguments regarding linear Fokker-Plank equations, see, for example, [39] and [56]. This leads one to guess that a Lyapunov functional for (7.3) would be given by

$$\mathcal{E}[u](t) = \int_{-\infty}^{+\infty} u(x,t)e^{-\int_{-\infty}^{x} u(z,t)dz}\log\left[\frac{u(x,t)e^{-\int_{-\infty}^{x} u(z,t)dz}}{e^{-W(x)}}\right]dx, \tag{7.8}$$

with $W(x) = 2\log\left[\cosh(\frac{c}{2}x)\right]$. If we formally differentiate $\mathcal{E}$, we obtain

$$\dot{\mathcal{E}}(t) = -\int_{-\infty}^{+\infty} u(x,t)e^{-\int_{-\infty}^{x} u(z,t)dz}\left(\partial_x \log\left[\frac{u(x,t)e^{-\int_{-\infty}^{x} u(z,t)dz}}{e^{-W(x)}}\right]\right)^2 dx. \tag{7.9}$$

This will be negative as long as $u(x,t) \geq 0$. (It turns out that this requirement is not necessary for the proof of global stability, as we will see below.) Therefore, we see that $\mathcal{E}$ is, in fact, a Lyapunov functional for equation (7.3).

To make this argument rigorous, we will apply LaSalle's Invariance principle, theorem 7.1.6. The maximal invariant set, M, in the statement of the theorem will be the one-parameter family of translates of the wave. We note that the following argument is similar to that of [26], in which the global stability of vortex solutions of the two-dimensional Navier-Stokes equations was studied.

7.2.2 Properties of Solutions in $L^1(\mathbb{R})$

First, we study the equation (7.3) in $L^1(\mathbb{R})$. Properties of the solution in this space will be used to remove the positivity requirement in equation (7.9).

The linear operator, $\mathcal{A} = \partial_x^2 + c\tanh(\frac{c}{2}x)\partial_x + \frac{c^2}{2}\text{sech}^2(\frac{c}{2}x)$, is the generator of an analytic semigroup in L^1. By using the theory of fractional Banach spaces in [31], we may conclude that solutions to equation (7.3) exist at least locally in time in L^1. To see that solutions exist globally as well, we will show that $||u(t)||_{L^1}$ is decreasing in time. Before doing so, we will need the following proposition about the positivity of solutions:

Lemma 7.2.1 *Given initial data $u_0 \in X$ such that $u_0(x) \geq 0$, the corresponding solution to equation (7.3) satisfies $u(x,t) > 0$ for all $x \in \mathbb{R}$ and $t > 0$.*

Proof The statement follows by applying the maximum principle, theorem 7 in chapter 3, section 3 of [46], to equation (7.5) and noting that the Cole-Hopf transformation (7.4) preserves positivity. We remark that the maximum principle could also be directly applied to equation (7.3), due to the fact that the nonlinearity is a perfect derivative and, hence, will vanish at any maximum point. □

Consider the function $\Phi : L^1(\mathbb{R}) \to \mathbb{R}^+$ defined by

$$\Phi(u) = \int |u(\xi)| d\xi \tag{7.10}$$

and define

$$\Sigma = \left\{ u \in L^1(\mathbb{R}) : \int |u(\xi)| d\xi = |\int u(\xi) d\xi| \right\}, \tag{7.11}$$

the set of all functions that are almost everywhere of the same sign. Note that, by lemma 7.2.1, this set is positively invariant under the flow of (7.3). We also note that for any initial data in L^1,

$$\int_{\mathbb{R}} u(\xi,t) d\xi = \int_{\mathbb{R}} u(\xi,0) d\xi. \tag{7.12}$$

Proposition 7.2.2 *Let $u_0 \in L^1(\mathbb{R})$. The corresponding solution to equation (7.3) satisfies $u \in C^0([0,\infty), L^1(\mathbb{R}))$. In addition, $\Phi[u](t) \leq \Phi[u_0]$ for all $t \geq 0$, and $\Phi[u](t) = \Phi[u_0]$ for all $t \geq 0$ if and only if $u_0 \in \Sigma$.*

Proof The proof is similar to that of Lemma 3.1 in [26]. If $u_0 \in \Sigma$, then $u(t) \in \Sigma$ for all $t \geq 0$. Thus we have

$$\Phi(u(t)) = \left| \int_{\mathbb{R}} u(\xi,t) d\xi \right| = \left| \int_{\mathbb{R}} u(\xi,0) d\xi \right| = \Phi(u_0).$$

Suppose now that $u_0 \notin \Sigma$. Then $u_0 = u_0^+ - u_0^-$, where

$$u_0^+(\xi) = \max(u_0(\xi),0) \geq 0, \quad u_0^-(\xi) = -\min(u_0(\xi),0) \geq 0.$$

By assumption, both u_0^+ and u_0^- are positive on a set of positive Lebesgue measure. Let

u_1 and u_2 be solutions of

$$\partial_t u_1 = \mathcal{A}u_1 - 2u_1\partial_\xi u_1,$$
$$\partial_t u_2 = \mathcal{A}u_2 + 2u_2\partial_\xi u_2 - 2\partial_\xi(u_1 u_2)$$

for $t \geq 0$ and with initial data $u_1(\xi, 0) = u_0^+$ and $u_2(\xi, 0) = u_0^-$. We see that

$$\int_{\mathbb{R}} u_1(\xi, t)d\xi = \int_{\mathbb{R}} u_0^+(\xi)d\xi, \quad \int_{\mathbb{R}} u_2(\xi, t)d\xi = \int_{\mathbb{R}} u_0^-(\xi)d\xi,$$

for all $t \geq 0$. Moreover, by a maximum principle argument similar to that of lemma 7.2.1, we have $u_1(\xi, t) > 0$ and $u_2(\xi, t) > 0$ for all $\xi \in \mathbb{R}$ and $t > 0$. Thus, one can verify that u_1, $u_2 \in C^0([0, \infty), L^1)$.

By construction, $u = u_1 - u_2$, and so

$$|u(\xi, t)| = |u_1(\xi, t) - u_2(\xi, t)| < u_1(\xi, t) + u_2(\xi, t),$$

for all $\xi \in \mathbb{R}$ and $t > 0$. Thus, we have that

$$\int_{\mathbb{R}} |u(\xi, t)|d\xi < \int_{\mathbb{R}} (u_1(\xi, t) + u_2(\xi, t))d\xi$$
$$= \int_{\mathbb{R}} u_0^+(\xi) + u_0^-(\xi)d\xi = \int_{\mathbb{R}} |u(\xi, 0)|d\xi,$$

for all $t \geq 0$. Therefore, $\Phi(u(t)) < \Phi(u(0))$ for all $t \geq 0$. □

This argument shows that Φ, which is just the L^1 norm of solutions, is a Lyapunov functional for equation (7.3). We next prove a result regarding the compactness of the omega limit sets of solutions.

Proposition 7.2.3 *Let $u_0 \in L^1$, and suppose $u \in C^0([0, \infty), L^1(\mathbb{R})$ is the corresponding solution of equation (7.3). Then the trajectory $\{u(t)\}_{t \geq 0}$ is relatively compact in L^1.*

Proof We will apply the Riesz criterion ([47], Theorem XIII.66) first to equation (7.5), and then show how the result can be extended to equation (7.3). In order to conclude that the trajectory $\{w(t)\}_{t \geq 0}$, corresponding to equation (7.5), is relatively compact in L^1, we must show

1. For any $\epsilon > 0$ there exists a $K \subset \mathbb{R}$ such that

$$\int_{\mathbb{R} \setminus K} |w(x, t)|dx \leq \epsilon \text{ for all } t \geq 0,$$

2. For any $\epsilon > 0$ there exists a $\delta > 0$ such that if $|z| < \delta$ then

$$\int_{\mathbb{R}} |w(x-z,t) - w(x,t)|dx \leq \epsilon \text{ for all } t \geq 0. \tag{7.13}$$

The solution to equation (7.5) can be written as

$$\begin{aligned}
w(x,t) = \partial_x &\left(\frac{2}{1+e^{-cx}} \frac{1}{\sqrt{4\pi t}} \int e^{-\frac{(y-(x+ct))^2}{4t}} \int_{-\infty}^{y} w_0(s)dsdy \right) \\
&+ \partial_x \left(\frac{2}{1+e^{+cx}} \frac{1}{\sqrt{4\pi t}} \int e^{-\frac{(y-(x-ct))^2}{4t}} \int_{-\infty}^{y} w_0(s)dsdy \right) \\
&= \left(\frac{2}{1+e^{-cx}} \frac{1}{\sqrt{4\pi t}} \int e^{-\frac{(y-(x+ct))^2}{4t}} w_0(y)dy + \frac{2}{1+e^{+cx}} \frac{1}{\sqrt{4\pi t}} \int e^{-\frac{(y-(x-ct))^2}{4t}} w_0(y)dy \right) \\
&+ \frac{c}{2}\text{sech}^2(\frac{c}{2}x) \frac{1}{\sqrt{4\pi t}} \int \left(\int_{-\infty}^{y} e^{-\frac{(z-(x+ct))^2}{4t}} dz \right) w_0(y)dy \\
&- \frac{c}{2}\text{sech}^2(\frac{c}{2}x) \frac{1}{\sqrt{4\pi t}} \int \left(\int_{-\infty}^{y} e^{-\frac{(z-(x-ct))^2}{4t}} dz \right) w_0(y)dy.
\end{aligned}$$

The terms involving a hyperbolic secant may be bounded by

$$||w_0||_{L^1} \int_{\mathbb{R}\setminus K} \text{sech}^2(\frac{c}{2}x)dx \leq \epsilon(K)||w_0||_{L^1}.$$

To deal with the remaining two terms, we will recombine them using the exponential factors, and then split up the integral:

$$\begin{aligned}
&\int_{\mathbb{R}\setminus K} |\frac{1}{\sqrt{4\pi t}} \text{sech}(\frac{c}{2}x) e^{-\frac{c^2}{4}t} \int_{|y|\geq R} e^{-\frac{(x-y)^2}{4t}} \cosh(\frac{c}{2}y) w_0(y)dy| \\
&\quad + \int_{\mathbb{R}\setminus K} |\frac{1}{\sqrt{4\pi t}} \text{sech}(\frac{c}{2}x) e^{-\frac{c^2}{4}t} \int_{|y|\leq R} e^{-\frac{(x-y)^2}{4t}} \cosh(\frac{c}{2}y) w_0(y)dy|dx.
\end{aligned}$$

In the first term, we will again split the integrand into two terms, which gives us

$$\begin{aligned}
&\int_{\mathbb{R}\setminus K} \frac{2}{1+e^{-cx}} \frac{1}{\sqrt{4\pi t}} \int_{|y|\geq R} e^{-\frac{(y-(x+ct))^2}{4t}} w_0(y)dy \\
&\quad + \int_{\mathbb{R}\setminus K} \frac{2}{1+e^{+cx}} \frac{1}{\sqrt{4\pi t}} \int_{|y|\geq R} e^{-\frac{(y-(x-ct))^2}{4t}} w_0(y)dy \\
&\leq C \int_{|y|\geq R} w_0(y)dy \leq \epsilon(R),
\end{aligned}$$

where we have used Fubini's theorem to first integrate the Gaussian with respect to x. If we now chose $K = \{|x| \leq 2R\}$, we may bound the integrals involving $|y| \leq R$ using the fact that

$$|x-y| \geq |x| - |y| \geq \frac{|x|}{2}, \text{ if } |x| \geq 2R \text{ and } |y| \leq R.$$

In addition, when $|x| \geq 2R$ and $|y| \leq R$, the term $\operatorname{sech}(\frac{c}{2}x)\cosh(\frac{c}{2}y)$ may be bounded independent of R. Hence, we have that

$$\begin{aligned}
&\int_{\mathbb{R}\setminus K} |\frac{1}{\sqrt{4\pi t}} \operatorname{sech}(\frac{c}{2}x) e^{-\frac{c^2}{4}t} \int_{|y|\leq R} e^{-\frac{(x-y)^2}{4t}} \cosh(\frac{c}{2}y) w_0(y) dy| dx \\
&\leq \int_{|x|\geq 2R} \int_{|y|\leq R} |\frac{1}{\sqrt{4\pi t}} e^{-\frac{c^2}{4}t} e^{-\frac{x^2}{16t}} w_0(y)| dy dx \\
&\leq ||w_0||_{L^1} \frac{e^{-\frac{c^2}{4}t}}{\sqrt{4\pi t}} \int_{|x|\geq 2R} e^{-\frac{x^2}{16t}} dx \leq \epsilon(R).
\end{aligned}$$

The final inequality follows by choosing R sufficiently large so that, for $0 < t \leq T$, the integral of the Gaussian is sufficiently small, while for $t > T$, the exponential factor $e^{-\frac{c^2}{4}t}$ is sufficiently small. Hence, we have proven item 1) of the Riesz criterion.

To prove item 2) of the Riesz criterion, we note that one can explicitly check that $|\partial_x w(x,t)| \leq K||w_0||_{L^1}$ for all $t \geq 0$. Therefore, we have that, if $\delta < 1$ is sufficiently small,

$$\begin{aligned}
\int_{|x|\geq R} |w(x-z,t) - w(x,t)| dx &\leq \int_{|x|\geq R} |w(x-z,t)| + \int_{|x|\geq R} |w(x,t)| dx \\
&\leq 2\int |x| \geq R-1 |w(x,t)| dx \\
&\leq 2\int |x| \geq R-1 |w(x,0)| dx \\
&\leq \frac{2\epsilon}{3}.
\end{aligned}$$

In addition,

$$\begin{aligned}
\int_{|x|\leq R} |w(x-z,t) - w(x,t)| dx &\leq 2R \sup_{|x|\leq R} |w(x-z,t) - w(x,t)| \\
&\leq 2R|z| \sup_{|x|\leq R-1} |w_x(x,t)| \\
&\leq K(R, \delta, ||w_0||_{L^1}) \\
&\leq \frac{\epsilon}{3}
\end{aligned}$$

This proves item 2) of the Riesz criterion, and we see that the trajectory $\{w(t)\}_{t\geq 0}$, corresponding to equation (7.5), is relatively compact in L^1. To extend this result to $\{u(t)\}_{t\geq 0}$, corresponding to equation (7.3), note that by inverting equation (7.4) we have

$$u(x,t) = \frac{w(x,t)}{1 - \int_{-\infty}^{x} w(s,t)ds}.$$

As a result, item 1) of the Riesz criterion can be written

$$\int_{\mathbb{R}\setminus K} |u(x,t)|dx \leq \frac{1}{e^{-\int |u(x,t)|}} \int_{\mathbb{R}\setminus K} |w(x,t)|dx.$$

Similarly, item 2) can be written

$$\int_{\mathbb{R}} |u(x-z,t) - u(x,t)|dx \leq \frac{C}{e^{-\int |u(x,t)|}} \int_{\mathbb{R}} |w(x-z,t) - w(x,t)|dx.$$

Therefore, $\{u(t)\}_{t\geq 0}$ is also relatively compact in L^1, which proves the proposition. □

We may now apply LaSalle's Invariance Principle (see theorem 7.1.6) to conclude that, if $u_0 \in L^1$, then $u(t)$ is asymptotically contained in the set E, the maximal invariant set such that $\dot{\Phi} = 0$. This is given exactly by $E = \Sigma$, and, thus, Σ is globally attracting. This is important because it allows us to remove the positivity requirement from equation (7.9) in the following manner.

Suppose we take $u_0 \in L^1$. By the preceding argument, the omega limit set of u_0 is contained in Σ, which means that any element of the omega limit set must be, almost everywhere, positive or negative. Consider now an element of the omega limit set. If $u(x) \geq 0$ almost everywhere, then the function $\mathcal{E}$, given in equation (7.8), can be used to show that the solution must approach a translate of the traveling wave. Suppose instead that $u(x) \leq 0$ almost everywhere. Then one must simply define $v = -u$ and consider the Lyapunov functional

$$\mathcal{E}[v](t) = \int_{-\infty}^{+\infty} v(x,t)e^{\int_{-\infty}^{x} v(z,t)dz} \log\left[\frac{v(x,t)e^{\int_{-\infty}^{x} v(z,t)dz}}{e^{-W(x)}}\right] dx.$$

One can then show that

$$\dot{\mathcal{E}}(t) = -\int_{-\infty}^{+\infty} v(x,t)e^{\int_{-\infty}^{x} v(z,t)dz} \left(\partial_x \log\left[\frac{v(x,t)e^{\int_{-\infty}^{x} v(z,t)dz}}{e^{-W(x)}}\right]\right)^2 dx,$$

which concludes the argument.

7.2.3 Global Stability of the traveling front

In order to complete the global stability argument, we must show that $\mathcal{E}$, given in equation (7.8), is a Lyapunov functional for equation (7.3) in an appropriately defined Banach space whose maximal invariant set is given by the family of translates of the traveling wave.

From equation (7.9) we see formally that the set of solutions for which $\dot{\mathcal{E}} = 0$ is given by the set of solutions that satisfy

$$0 = u_x + c\tanh(\frac{c}{2}x)u - u^2. \tag{7.14}$$

This family of solutions is given by

$$u_\infty^K(x) = \frac{\frac{c}{2}\text{sech}^2(\frac{c}{2}x)}{K - \tanh(\frac{c}{2}x)}; \quad |K| > 1, \tag{7.15}$$

which corresponds exactly to the family of translates of the wave. To see this, notice that

$$\begin{aligned}
\varphi(x) + u_\infty^K(x) &= \frac{c}{1+e^{cx}} + \frac{\frac{c}{2}\text{sech}^2(\frac{c}{2}x)}{K - \tanh(\frac{c}{2}x)} \\
&= \frac{\frac{c}{2}e^{-\frac{c}{2}x}\text{sech}(\frac{c}{2}x)\left(K - \tanh(\frac{c}{2}x)\right) + \frac{c}{2}\text{sech}^2(\frac{c}{2}x)}{K - \tanh(\frac{c}{2}x)} \\
&= \frac{\frac{c}{2}e^{-\frac{c}{2}x}\left(K - \tanh(\frac{c}{2}x)\right) + \frac{c}{2}\text{sech}(\frac{c}{2}x)}{K\cosh(\frac{c}{2}x) - \sinh(\frac{c}{2}x)} \\
&= \frac{\frac{c}{2}e^{-\frac{c}{2}x}\left(K - \tanh(\frac{c}{2}x)\right) + \frac{c}{2}\text{sech}(\frac{c}{2}x)}{\frac{1}{2}(K+1)e^{-\frac{c}{2}x}\left(1 + e^{c(\delta+x)}\right)} \\
&= \frac{c}{1+e^{c(\delta+x)}},
\end{aligned}$$

where $\delta = \frac{1}{c}\log(\frac{K-1}{K+1})$. Note also that $\int_{\mathbb{R}} u_\infty^K(x)d\xi = \log(\frac{K+1}{K-1})$.

The difficulty now lies in choosing the appropriate Banach space within which to work. In such a space, solutions to equation (7.3) must exist globally in time with relatively compact trajectories, and the function $\mathcal{E}$ must be continuous. Although the arguments of section 7.2.2 suggest that L^1 is a natural choice, it is difficult to prove continuity of $\mathcal{E}$ there. On the other hand, it is not too difficult to show that $\mathcal{E}$ is continuous in the algebraically weighted spaces $L^2(m)$, defined in equation (3.18), for $m \geq 3$. However, these spaces do not seem natural for the semigroup generated by the operator $\partial_x^2 + c\partial_x\tanh(\frac{c}{2}x) + \frac{c^2}{2}\text{sech}^2(\frac{c}{2}x)$. The completion of this argument is the subject of current work. We note that this problem illustrates an important point: sometimes one can find relatively strong evidence suggesting a certain result, but rigorously justifying it can be quite difficult!

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