Homework Assignment 1, Due Wednesday, Jan 27

1) For each of the following maps, construct the suspension and describe the space $\tilde{X}$.

   (a) $f : [0,1] \to [0,1]$, $f(x) = 1 - x$.

   (b) $f : S^1 \to S^1$, $f(z) = \bar{z}$.

2) Give an example of a metric space X, a homeomorphism $T : X \to X$, and a point $x \in X$ such that $\cup_{n=-\infty}^{\infty} T^n(x)$ is dense but $\cup_{n=0}^{\infty} T^n(x)$ is not dense. What about an example where X is compact?

3) Given any interval I, define the "measure" of I to be the length of I, and denote it by $\lambda(I)$. Similarly, if I is any countable union of disjoint intervals, define $\lambda(\cup_{i=1}^{\infty} I_i) = \sum_{i=1}^{\infty} \lambda(I_i)$. Prove that $\lambda(E_m^{-1}([a,b])) = \lambda([a,b])$ for all $[a,b] \subset [0,1]$, where E_m is the expanding map we discussed in class: $E_m : S^1 \to S^1$, where $x \mapsto mx$ mod 1.

4) Consider the Tent Map

$$T : [0,1] \to [0,1], \quad T(x) = \begin{cases} 2x & \text{if } 0 \le x \le 1/2 \\ 2 - 2x & \text{if } 1/2 \le x \le 1 \end{cases}$$

   (a) Sketch the graph of T, T^2, and a representative graph of T^n.

   (b) Use the graph of T^n to explain why T has exactly 2^n points of period n (n need not be minimal).

   (c) Prove the set of all periodic points is dense in $[0,1]$.

   (d) Find the number of points with least period $1, 2, 4$. Find the number of distinct orbits with least period $1, 2, 4$.

5) Show that a homeomorphism f of $\mathbb{R}/\mathbb{Z}$ of degree -1 has exactly two fixed points and that an interval bounded by these two points maps homeomorphically onto itself under $f \circ f$. (This is problem 4.6.6 of Milnor. You can use the results of problem 4.6.5. There it is shown that, given any lift F, there exists a unique n, called the degree of f, such that $F(t+1) = F(t) + n$ for all t.)

6) Symbolic dynamics for the m-tupling map:

   (a) For any $m \ge 2$, $m \in \mathbb{Z}$, let $E_m : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$, $x \mapsto mx$ mod1. Use intervals of the form

$$I_\alpha = \left[\frac{\alpha}{m}, \frac{\alpha+1}{m}\right]$$

   to cover the circle $\mathbb{R}/\mathbb{Z}$. Given any infinite sequence $\alpha_0, \alpha_1, \ldots$, where $\alpha_i \in \{0, 1, \ldots, m-1\}$, show that there is one and only one number

$$x_0 = 0.\alpha_0\alpha_1\alpha_2\ldots(\text{base } m) = \sum_{i=0}^{\infty} \frac{\alpha_i}{m^{i+1}}$$

   in $\mathbb{R}/\mathbb{Z}$ with the property that the orbit of x under E_m satisfies $E_m^k(x) \in I_{\alpha_k}$ for all k.

   (b) Use this information to show that the shift map $\sigma_m : \{0, 1, \ldots, m-1\}^{\mathbb{N}} \to \{0, 1, \ldots, m-1\}^{\mathbb{N}}$ is semi-conjugate to the m-tupling map. Why can it not be a conjugacy?