1) Let $f : X \to X$ be continuous and $X$ compact. If $A_{1,2}$ are two distinct trapped attractors (i.e., at least one $x \in A_i$ such that $x \notin A_j$, for $i, j = 1, 2, i \neq j$) for the map $f$, show that $A_1 \cap A_2 = \emptyset$.

2) Prove that a circle rotation is ergodic with respect to Lebesgue measure if and only if $\alpha$ is irrational using the following facts (you can use these without proving them):

   (a) A function $h : X \to \mathbb{R}$ is called essentially invariant for the map $f$ if $\mu(\{x \in X : h(f(x)) \neq h(x)\}) = 0$. The map $f$ is ergodic with respect to $\mu$ if any only if every essentially invariant function $h \in L^p(X, \mu)$, for some $0 < p \leq \infty$, is constant.

   (b) Given any $f \in L^2(X, \text{Leb})$, where $\text{Leb}$ is Lebesgue measure (so this is just normal integration), the Fourier series $\sum_{n=-\infty}^{+\infty} a_n e^{2\pi i nx}$ converges to $f$ in the $L^2$ norm and the coefficients are uniquely determined.

   (Hint: suppose $f$ is essentially invariant. What does this imply about its Fourier coefficients?)

3) The Borel Normal Number Theorem states that Lebesgue-almost all $x \in [0, 1)$ are normal to base 2, meaning that the frequency of 1’s in their binary expansion is $1/2$. Prove this using the doubling map and the Birkhoff ergodic theorem. Note: you may use the fact that Lebesgue is ergodic for the doubling map.

4) Let $f : \mathbb{T}^2 \to \mathbb{T}^2$ be the diffeomorphism inducted by the matrix

   $$A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.$$ 

   Form a Markov partition with the three rectangles by using the line segments $[a, b]_s$ and $[g, c]_u$, where $g$ is determined by extending the unstable manifold of the origin through the origin so that it terminates at a point $g \in [a, 0]_s$. Thus, $0$ is within the unstable segment $[g, c]_u$. Determine the transition matrix $B$ for this partition. Determine the three eigenvalues for the transition matrix. How do the eigenvalues compare with those of $A$?