Homework Assignment 4, Due Date Monday, April 26

1) Let $f : X \to X$ be continuous and X compact. If $A_{1,2}$ are two distinct trapped attractors (ie $\exists$ at least one $x \in A_i$ such that $x \notin A_j$, for $i, j = 1, 2$, $i \neq j$) for the map f, show that $A_1 \cap A_2 = \emptyset$.

2) Prove that a circle rotation is ergodic with respect to Lebesgue measure if and only if α is irrational using the following facts (you can use these without proving them):

   (a) A function $h : X \to \mathbb{R}$ is called essentially invariant for the map f if $\mu(\{x \in X : h(f(x)) \neq h(x)\}) = 0$. The map f is ergodic with respect to μ if any only if every essentially invariant function $h \in L^p(X, \mu)$, for some $0 < p \leq \infty$, is constant.

   (b) Given any $f \in L^2(X, \mathrm{Leb})$, where Leb is Lebesgue measure (so this is just normal integration), the Fourier series $\sum_{n=-\infty}^{+\infty} a_n e^{2n\pi i x}$ converges to f in the L^2 norm and the coefficients are uniquely determined.

   (Hint: suppose f is essentially invariant. What does this imply about its Fourier coefficients?)

3) The Borel Normal Number Theorem states that Lebesgue-almost all $x \in [0, 1)$ are normal to base 2, meaning that the frequency of 1's in their binary expansion is $1/2$. Prove this using the doubling map and the Birkhoff ergodic theorem. Note: you may use the fact that Lebesgue is ergodic for the doubling map.

4) Let $f : \mathbb{T}^2 \to \mathbb{T}^2$ be the diffeomorphism inducted by the matrix

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.$$

   Form a Markov partition with the three rectangles by using the line segments $[\mathbf{a}, \mathbf{b}]_s$ and $[\mathbf{g}, \mathbf{c}]_u$, where $\mathbf{g}$ is determined by extending the unstable manifold of the origin through the origin so that it terminates at a point $\mathbf{g} \in [\mathbf{a}, \mathbf{0}]_s$. Thus, $\mathbf{0}$ is within the unstable segment $[\mathbf{g}, \mathbf{c}]_u$. Determine the transition matrix B for this partition. Determine the three eigenvalues for the transition matrix. How do the eigenvalues compare with those of A?