Solutions to Calc I & II review – MA 225 B1 – Spring 2011

If you got (m)any of these wrong, don't worry! It has been at least a month since you took Calc II, and longer since you had Calc I. Just be aware that much of what we will do in this class will rely on material from Calc I & II. So, if we starting using any such material and you don't remember it, please go back and review your notes from your previous calculus classes!

Question 1

(i) Using the product and chain rules, $f'(x) = \cos(x)e^{1/x} - (1/x^2)\sin(x)e^{1/x}$.

(ii) Using the u-substitution $u = 2x+1$ and changing the limits of integration accordingly, $\int_1^\infty 1/(2x+1)^3 \mathrm{d}x = (1/2)\int_3^\infty u^{-3}\mathrm{d}u = (-1/4)\lim_{b\to\infty} u^{-2}|_3^b = 1/36$.

Question 2

(i) $f'(b)$ is the slope of the tangent line to the curve at $x = b$. Since $x = b$ is a local maximum, $f'(b) = 0$.

(ii) The integral is the area under the curve between a and c, where anything below the x-axis counts as negative area, and anything above it counts as positive area. It looks like there is more positive area than negative area, so the integral will be positive.

(iii) $f''(b)$ represents the concavity of f at b. Since the curve is concave down at $x = b$, $f''(b) < 0$. Alternatively, $f''(b)$ is the derivative of $f'(x)$ evaluated at $x = b$. Since $f'(x) > 0$ for $x < b$ and $f'(x) < 0$ for $x > b$, we see that $f'(x)$ is decreasing at $x = b$. Hence, $f''(b) < 0$.

Question 3

(i)

$$f'(b) = \lim_{h\to 0}\frac{f(b+h) - f(b)}{h}$$

The quantity inside the limit represents the slope of the line connecting the points $(b, f(b))$ and $(b+h, f(b+h))$. As $h \to 0$, these lines approximate the tangent line, and so the above limit converges to the slope of the tangent line at $x = b$.

(ii)

$$\int_a^c f(x)\mathrm{d}x = \lim_{n\to\infty}\sum_{i=1}^n f(x_i)\Delta x,$$

where the interval $[a, c]$ has been divided into n subintervals of length Δx and x_i is any point within the i-th subinterval. For fixed n, this is a sum of areas of rectangles with heights $f(x_i)$ and each having the same width Δx. As $n \to \infty$ and the rectangles get more and more narrow, this converges to the area under the curve. Note that if $f(x_i) < 0$, then the area of the corresponding rectangle gets a minus sign in front of it, so any area below the x-axis gets counted negatively.

Question 4 Note that by setting $x = 1 + y^2$ and $x = y + 3$ equal to each other, we solve the equation $y^2 - y - 2 = (y-2)(y+1) = 0$ to see that these curves intersect at the points $(2, -1)$ and $(5, 2)$. If we slice through the solid at any fixed value of y, we will get an annulus whose inner radius is given by $1+y^2$ and whose outer radius is given by $y+3$. Thus, the area of that annulus will be $\pi[(y+3)^2 - (1+y^2)^2]$. We then need to sum these up for y in the interval $[-1, 2]$, which gives

$$V = \pi \int_{-1}^{2} [(y+3)^2 - (1+y^2)^2] \mathrm{d}y.$$

Question 5 If we let $h(s)$ be the height of the ballon, then $h'(s) = 5$ for all s and $h(0) = 45$. Hence, $h(s) = 5s + 45$. If $l(s)$ is the distance between the boy and the spot on the road directly under the balloon, then $l'(s) = 15$ for all s and $l(0) = 0$. Hence, $l(s) = 15s$. Drawing a picture we see that the distance between the boy and the balloon is the hypotenuse of a right triangle with sides $l(s)$ and $h(s)$, so the distance is given by $d(s) = \sqrt{l^2(s) + h^2(s)}$. Thus,

$$d'(s) = \frac{l(s)l'(s) + h(s)h'(s)}{\sqrt{l^2(s) + h^2(s)}} \quad \Rightarrow \quad d'(3) = \frac{45 \cdot 15 + 60 \cdot 5}{\sqrt{45^2 + 60^2}} = 13\text{ft/s}.$$