Solutions to Midterm 1a – MA 225 B1 – Spring 2011

Question 1

(i) $\mathbf{b} \times \mathbf{c}$ is a vector, so we can take its dot product with $\mathbf{a}$. Since the dot product of two vectors is a scalar, this quantity is a scalar.

(ii) Both $(\mathbf{a} \cdot \mathbf{b})$ and $(\mathbf{c} \cdot \mathbf{d})$ are scalars, so it does not make sense to take their cross product. Hence, this quantity is not meaningful.

Question 2 We are given a point on the plane, so we need to find a normal vector. We can do this by finding two vectors in the plane, and taking their cross product. One such vector is the direction vector of the line, which is $\mathbf{v}_1 = \langle 1, 1/2, 1/3 \rangle$. Another vector can be found by taking a point on the line, for example $Q = (0,0,0)$, and using $\mathbf{v}_2 = \langle 1, -1, 2 \rangle$. Thus, we have $\mathbf{n} = \mathbf{v}_1 \times \mathbf{v}_2 = \langle 4/3, -5/3, -3/2 \rangle$, and the equation for the plane is

$$\frac{4}{3}(x-1) - \frac{5}{3}(y+1) - \frac{3}{2}(z-2) = 0.$$

Question 3 The equations $\rho = 1$ and $\rho = 4$ represent the spheres of radii one and four, respectively. The inequality involving ϕ implies we only take the bottom half (like the southern hemisphere) of each. Thus, the solid is the bottom half of a sphere of radius four, with the bottom half of the sphere of radius one cut out from inside it. This solid includes all its boundaries, because none of the inequalities are strict inequalities.

Question 4 The derivative of this curve is $\mathbf{r}'(t) = \langle e^t, e^t \sin t + e^t \cos t, -e^t \sin t + e^t \cos t \rangle$, and so

$$s(t) = \int_0^t |\mathbf{r}'(u)| \mathrm{d}u = \int_0^t \sqrt{e^{2u}(1 + 2\sin^2 u + 2\cos^2 u)} \mathrm{d}u = \sqrt{3} \int_0^t e^u \mathrm{d}u = \sqrt{3}(e^t - 1).$$

Hence, $t = \ln[1 + s/\sqrt{3}]$, and

$$\mathrm{r}(s) = \langle (1 + s/\sqrt{3}), (1 + s/\sqrt{3}) \sin \ln[1 + s/\sqrt{3}], (1 + s/\sqrt{3}) \cos \ln[1 + s/\sqrt{3}] \rangle.$$

Question 5

(i) The z coordinate is fixed at 2, so the entire curve lies in the plane parallel to the xy plane with height 2. Since $x = t^2 = y^2$, this curve is a parabola that opens up along the positive x axis. Also, $\mathbf{r}(0) = \langle 0, 0, 2 \rangle$ and $\mathbf{r}(1) = \langle 1, 1, 2 \rangle$, so the curve is moving from the region where x is positive and y is negative to the region where x is positive and y is positive.

(ii) This point corresponds to $t = -2$. The derivative at that point is $\mathbf{r}'(-2) = \langle 2(-2), 1, 0 \rangle$. Thus, the line is $\mathbf{r}_0 + t\mathbf{v} = \langle 4, -2, 2 \rangle + t\langle -4, 1, 0 \rangle$.

Question 6 We're told $\alpha = \pi/6$. The maximum hight is reached when the derivative of the second component is zero, so when $v_0 \sin \pi/6 - gt = 0$, or at the time $t^* = \frac{v_0}{g} \sin \pi/6$. If we evaluate the second component at this time, we must get 500. Hence,

$$500 = v_0 \sin \pi/6 \left(\frac{v_0}{g} \sin \pi/6 \right) - \frac{1}{2} g \left(\frac{v_0}{g} \sin \pi/6 \right)^2 = \frac{v_0^2}{2g} \sin^2 \pi/6.$$

Thus, the initial speed is $v_0 = 10\sqrt{10g}/\sin(\pi/6)$.

Question 7

(i) If we set $x = t$ we find $\mathbf{r}(t) = \langle t, t^2, 4t^2 + t^4 \rangle$.

(ii) Notice that $x^2 + z^2 = s^2 = y$. This equation describes a paraboloid that starts at the origin and opens along the positive y axis. (Traces parallel to the xz plane are circles.)

Question 8 We compute

$$\text{proj}_{\mathbf{b}} \mathbf{a} = \left(\frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{b}|^2} \right) \mathbf{b} = \frac{1}{2} \langle -1, 0, 1 \rangle.$$

Solutions to Midterm 1b – MA 225 B1 – Spring 2011

Question 1 We compute

$$\mathrm{proj}_{\mathbf{b}}\mathbf{a} = \left(\frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{b}|^2}\right)\mathbf{b} = \frac{6}{5}\langle 2, 1, 0\rangle.$$

Question 2 We are given a point on the plane, so we need to find a normal vector. We can do this by finding two vectors in the plane, and taking their cross product. One such vector is the direction vector of the line, which is $\mathbf{v}_1 = \langle 1/2, 1, 1/4\rangle$. Another vector can be found by taking a point on the line, for example $Q = (0, 0, 0)$, and using $\mathbf{v}_2 = \langle 2, 0, -1\rangle$. Thus, we have $\mathbf{n} = \mathbf{v}_1 \times \mathbf{v}_2 = \langle -1, 1, -2\rangle$, and the equation for the plane is

$$-(x-2) + y - 2(z+1) = 0.$$

Question 3

(i) The x coordinate is fixed at 2, so the entire curve lies in the plane parallel to the yz plane at postive 2 units on the x axis. Since $z = t^2 = y^2$, this curve is a parabola that opens up along the positive z axis. Also, $\mathbf{r}(0) = \langle 2, 0, 0\rangle$ and $\mathbf{r}(1) = \langle 2, 1, 1\rangle$, so the curve is moving from the region where z is positive and y is negative to the region where z is positive and y is positive.

(ii) This point corresponds to $t = -3$. The derivative at that point is $\mathbf{r}'(-2) = \langle 0, 1, 2(-3)\rangle$. Thus, the line is $\mathbf{r}_0 + t\mathbf{v} = \langle 2, -3, 9\rangle + t\langle 0, 1, -6\rangle$.

Question 4 The equations $\rho = 2$ and $\rho = 5$ represent the spheres of radii two and five, respectively. The inequality involving ϕ implies we only take the top half (like the northern hemisphere) of each. Thus, the solid is the top half of a sphere of radius five, with the top half of the sphere of radius two cut out from inside it. This solid includes all its boundaries, because none of the inequalities are strict inequalities.

Question 5 The derivative of this curve is $\mathbf{r}'(t) = \langle e^t \sin t + e^t \cos t, e^t, -e^t \sin t + e^t \cos t\rangle$, and so

$$s(t) = \int_0^t |\mathbf{r}'(u)|\mathrm{d}u = \int_0^t \sqrt{e^{2u}(1 + 2\sin^2 u + 2\cos^2 u)}\mathrm{d}u = \sqrt{3}\int_0^t e^u \mathrm{d}u = \sqrt{3}(e^t - 1).$$

Hence, $t = \ln[1 + s/\sqrt{3}]$, and

$$\mathrm{r}(s) = \langle (1 + s/\sqrt{3})\sin\ln[1 + s/\sqrt{3}], (1 + s/\sqrt{3}), (1 + s/\sqrt{3})\cos\ln[1 + s/\sqrt{3}]\rangle.$$

Question 6

(i) If we set $y = t$ we find $\mathbf{r}(t) = \langle t^2, t, 2t^4 + 4t^2\rangle$.

(ii) Notice that $y^2 + z^2 = t^2 = x$. This equation describes a paraboloid that starts at the origin and opens along the positive x axis. (Traces parallel to the yz plane are circles.)

Question 7 We're told $\alpha = \pi/4$. The maximum hight is reached when the derivative of the second component is zero, so when $v_0 \sin \pi/4 - gt = 0$, or at the time $t^* = \frac{v_0}{g} \sin \pi/4$. If we evaluate the second component at this time, we must get 200. Hence,

$$200 = v_0 \sin \pi/4 \left(\frac{v_0}{g} \sin \pi/4 \right) - \frac{1}{2} g \left(\frac{v_0}{g} \sin \pi/4 \right)^2 = \frac{v_0^2}{2g} \sin^2 \pi/4.$$

Thus, the initial speed is $v_0 = 20\sqrt{g}/\sin(\pi/4)$.

Question 8

(i) $\mathbf{a} \cdot \mathbf{b}$ is a scalar, so we cannot take its cross product with $\mathbf{c}$. Hence, this quantity is not meaningful.

(ii) Both $(\mathbf{a} \times \mathbf{b})$ and $(\mathbf{c} \times \mathbf{d})$ are scalars, so it does make sense to take their cross product, which produces a vector.