Solutions to Midterm 2a – MA 225 B1 – Spring 2011

Question 1 Differentiating both sides of the equation with respect to y, we obtain

$$z + yz_y = z_y/(x+z).$$

Solving this equation for z_y, we find

$$\frac{\partial z}{\partial y} = \frac{z(x+z)}{1 - y(x+z)}.$$

Question 2 The linearization is given by

$$g(x,y) \approx g(x_0,y_0) + g_x(x_0,y_0)(x-x_0) + g_y(x_0,y_0)(y-y_0).$$

Since $(x_0,y_0) = (1,1)$, $g_x(x,y) = (2xye^{xy} + x^2y^2e^{xy})$ and $g_y(x,y) = x^2e^{xy} + x^3ye^{xy}$, we have $g(1,1) = e$, $g_x(1,1) = 3e$ and $g_y(1,1) = 2e$. Hence, $g(0.9, 1.1) \approx e - 0.1(3e) + 0.1(2e) = (0.9)e$.

Question 3 Since $f_x = 4 - 2x$ and $f_y = 6 - 2y$, the critical point is $(2,3)$, and $f(2,3) = 13$. To check the boundary, we check the interior of each of the four sides, and then we check the four corners. We get $f_x(x,5) = -2x + 4$, which is zero when $x = 2$, and $f(2,5) = 9$. Also, $f_x(x,0) = 4 - 2x$, and $f(2,0) = 4$. Also, $f_y(4,y) = 6 - 2y$, and $f(4,3) = 9$. Also, $f_y(0,y) = 6 - 2y$, and $f(0,3) = 9$. For the corners, $f(0,0) = 0$, $f(0,5) = 5$, $f(4,0) = 0$, and $f(4,5) = 5$. Hence, the maximum is at $(2,3)$, where $f(2,3) = 13$, and the two minima are at $(0,0)$ and $(4,0)$, where $f(0,0) = f(4,0) = 0$.

Question 4 By the properties of the directional derivative, the maximum rate of change is $|\nabla g(0,0)|$ and it occurs in the direction of $\nabla g(0,0)$. Since $\nabla g = \langle e^{-y} - ye^{-x}, -xe^{-y} + e^{-x}\rangle$, we have $\nabla g(0,0) = \langle 1, 1\rangle$ and $|\nabla g(0,0)| = \sqrt{2}$.

Question 5 We have that the surface area is equation to

$$\begin{aligned}
\int_0^1 \int_0^\pi |\mathbf{r}_u \times \mathbf{r}_v| dv du &= \int_0^1 \int_0^\pi |\langle \cos v, \sin v, 0\rangle \times \langle -u\sin v, u\cos v, 1\rangle| dv du \\
&= \int_0^1 \int_0^\pi |\langle \sin v, -\cos v, u\rangle| dv du = \int_0^1 \int_0^\pi \sqrt{\sin^2 v + \cos^2 v + u^2} dv du \\
&= \int_0^1 \int_0^\pi \sqrt{1 + u^2} dv du.
\end{aligned}$$

Question 6 The constraint is $g(x,y) = x^3 + y^3 - 16 = 0$. The equation $\nabla f = \lambda \nabla g$ is $\langle ye^{xy}, xe^{xy}\rangle = \lambda\langle 3x^2, 3y^2\rangle$. Thus,

$$ye^{xy} = 3\lambda x^2, \quad xe^{xy} = 3\lambda y^2, \quad \Rightarrow \quad \frac{3\lambda x^2}{y} = \frac{3\lambda y^2}{x}.$$

Manipulating this equation, we find $x^3 = y^3$ (or $\lambda = 0$, but that implies $(x,y) = (0,0)$, which doesn't satisfy the constraint). Plugging this into the constraint, we have $2x^3 = 16$, or $x = 2$. This in turn implies $y = 2$. This must correspond to a maximum, rather than a minimum, because one can find values of (x, y) that satisfy the constraint and make f arbitrarily close to zero. Thus, $f(2, 2) = e^4$ is the maximum of f, subject to the constraint. (Alternatively one can check it is a maximum using the second derivative test.)

Question 7 We have

$$\begin{aligned} \int_0^1 \int_0^{x^2} \frac{y}{1+x^5} \mathrm{d}y\mathrm{d}x &= \int_0^1 \frac{y^2}{2(1+x^5)} \Big|_{y=0}^{y=x^2} \mathrm{d}x \\ &= \int_0^1 \frac{x^4}{2(1+x^5)} \mathrm{d}x = \frac{1}{10} \ln(1+x^5)\Big|_0^1 = \frac{1}{10} \ln 2. \end{aligned}$$

Question 8 Drawing the projection of the planes in the xy-plane, we see that the region of integration, D, is bounded by the lines $x = 0$, $y = x$, and $y = -x + 2$. This gives

$$\int_0^1 \int_x^{-x+2} x \ \mathrm{d}y\mathrm{d}x.$$

Question 9 This integral implies that the region of integration in the xy-plane is bounded by the lines $y = x$, $y = 0$, and $x = \sqrt{\pi}$. Thus, we switch the order of integration by writing

$$\int_0^{\sqrt{\pi}} \int_0^x \cos(x^2)\mathrm{d}y\mathrm{d}x.$$

Question 10 The upper limit of integration in y is when $y = \sqrt{2x - x^2}$, or $y^2 + x^2 = 2x$. By completing the square, one can see that this is a circle of radius one with center $(1, 0)$. Since $0 \le y \le \sqrt{2x - x^2}$, we only take the top half of the circle, so $0 \le \theta \le \pi/2$. Converting the equation for the circle to polar coordinates, we have $r = 2\cos\theta$, and so $0 \le r \le 2\cos\theta$. Hence, since the integrand is r and we get an extra factor of r from the change of variables formula,

$$\int_0^{\pi/2} \int_0^{2\cos\theta} r^2 \mathrm{d}r\mathrm{d}\theta.$$

Solutions to Midterm 2b – MA 225 B1 – Spring 2011

Question 1 We have

$$\int_0^1 \int_0^{y^2} \frac{3x}{1+y^5} \mathrm{d}x\mathrm{d}xy = \int_0^1 \frac{3x^2}{2(1+y^5)}\Big|_{x=0}^{x=y^2} \mathrm{d}y$$

$$= \int_0^1 \frac{3y^4}{2(1+y^5)}\mathrm{d}y = \frac{3}{10}\ln(1+y^5)\Big|_0^1 = \frac{3}{10}\ln 2.$$

Question 2 This integral implies that the region of integration in the xy-plane is bounded by the lines $y = x$, $y = \sqrt{\pi}$, and $x = 0$. Thus, we switch the order of integration by writing

$$\int_0^{\sqrt{\pi}} \int_0^{y} \cos(y^2)\mathrm{d}x\mathrm{d}y.$$

Question 3 The constraint is $g(x, y) = x^3 + y^3 - 54 = 0$. The equation $\nabla f = \lambda \nabla g$ is $\langle ye^{xy}, xe^{xy}\rangle = \lambda\langle 3x^2, 3y^2\rangle$. Thus,

$$ye^{xy} = 3\lambda x^2, \quad xe^{xy} = 3\lambda y^2, \quad \Rightarrow \quad \frac{3\lambda x^2}{y} = \frac{3\lambda y^2}{x}.$$

Manipulating this equation, we find $x^3 = y^3$ (or $\lambda = 0$, but that implies $(x, y) = (0, 0)$, which doesn't satisfy the constraint). Plugging this into the constraint, we have $2x^3 = 54$, or $x = 3$. This in turn implies $y = 3$. This must correspond to a maximum, rather than a minimum, because one can find values of (x, y) that satisfy the constraint and make f arbitrarily close to zero. Thus, $f(3, 3) = e^9$ is the maximum of f, subject to the constraint. (Alternatively one can check it is a maximum using the second derivative test.)

Question 4 By the properties of the directional derivative, the maximum rate of change is $|\nabla g(1, 1)|$ and it occurs in the direction of $\nabla g(1, 1)$. Since $\nabla g = \langle e^{-y} + ye^{-x}, -xe^{-y} - e^{-x}\rangle$, we have $\nabla g(1, 1) = \langle 2e^{-1}, -2e^{-1}\rangle$ and $|\nabla g(1, 1)| = \sqrt{8e^{-2}} = 2e^{-1}\sqrt{2}$.

Question 5 The linearization is given by

$$g(x, y) \approx g(x_0, y_0) + g_x(x_0, y_0)(x - x_0) + g_y(x_0, y_0)(y - y_0).$$

Since $(x_0, y_0) = (1, 1)$, $g_x(x, y) = (y^2e^{xy} + xy^3e^{xy})$ and $g_y(x, y) = 2yxe^{xy} + x^2y^2e^{xy}$, we have $g(1, 1) = e$, $g_x(1, 1) = 2e$ and $g_y(1, 1) = 3e$. Hence, $g(0.9, 1.1) \approx e - 0.1(2e) + 0.1(3e) = (1.1)e$.

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Question 8 Drawing the projection of the planes in the xy-plane, we see that the region of integration, D, is bounded by the lines $x = 0$, $y = x$, and $y = -x + 4$. This gives

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&= \int_0^1 \int_0^\pi |\langle v, -\cos u, \sin u\rangle| \mathrm{d}u\mathrm{d}v = \int_0^1 \int_0^\pi \sqrt{\sin^2 u + \cos^2 u + v^2} \mathrm{d}u\mathrm{d}v \\
&= \int_0^1 \int_0^\pi \sqrt{1 + v^2} \mathrm{d}u\mathrm{d}v.
\end{aligned}$$