Solutions to Practice Final – MA 225 B1 – Spring 2011

(i) Using the change of variables $u = 3x$ and $v = 2y$, we find that the ellipse transforms into a circle. The Jacobian of this transformation is $1/6$, so, using polar coordinates in the new variables we have

$$\iint_R \sin(9x^2 + 4y^2)\mathrm{d}A = \iint_D \sin(u^2 + v^2)\frac{1}{6}\mathrm{d}A = \frac{1}{6}\int_0^{2\pi}\int_0^1 r\sin(r^2)\mathrm{d}r\mathrm{d}\theta = \frac{\pi}{6}(1 - \cos(1)).$$

(ii) Since div$\mathbf{F}$ is a scalar, it does not make sense to take the cross product of this quantity and the vector $\mathbf{b}$. Thus, it is not meaningful.

(iii) The x and y components of the cylinder can be parameterized using $x = \cos t$, $y = \sin t$, $0 \le t \le 2\pi$. Thus, $z = \cos t + 3$, and so we find

$$\mathbf{r}(t) = \langle \cos t, \sin t, \cos t + 3\rangle, \qquad 0 \le t \le 2\pi.$$

(iv) The direction vectors of the two lines are $\mathbf{v}_1 = \langle 2, 3, -1\rangle$ and $\mathbf{v}_2 = \langle 2, 1, 7\rangle$. These vectors are not parallel (in fact they are orthogonal, since their dot product is zero), and so the lines are not parallel. To see if they intersect, we write a parameterization of the second line, $x = 2s + 3$, $y = s - 1$, $z = 7s + 1$, and see if we can solve both sets of equation simultaneously. Looking at the x and y equations only, we find $t = -1$ and $s = -2$, but these values do not make the z coordinates equation, and so there is no point of intersection. Thus, the lines are skew.

(v) We will apply the divergence theorem. The plane forming the top of the tetrahedron is $x+y+z = 1$, and so

$$\begin{aligned}
\iint_S \mathbf{F}\cdot\mathrm{d}\mathbf{S} &= \iiint_E \mathrm{div}\mathbf{F}\mathrm{d}V = \int_0^1\int_0^{-x+1}\int_0^{1-x-y}(-1)\mathrm{d}z\mathrm{d}y\mathrm{d}x = \int_0^1\int_0^{-x+1}(x+y-1)\mathrm{d}y\mathrm{d}x \\
&= \int_0^1 [(x-1)(-x+1) + \frac{1}{2}(-x+1)^2]\mathrm{d}x = -\int_0^1 \frac{1}{2}(-x+1)^2\mathrm{d}x \\
&= \frac{1}{6}(1-x)^3|_0^1 = -1/6
\end{aligned}$$

(vi) This is just a triple integral over the entire sphere of radius 3, and so we convert to spherical coordinates to find

$$\int_0^{2\pi}\int_0^{\pi}\int_0^3 \rho^5 \sin\phi\cos\phi\mathrm{d}\rho\mathrm{d}\phi\mathrm{d}\theta = (2\pi)(243/2)(0) = 0.$$

(vii) Since the plane doesn't intersect the xz-plane, it must be parallel to it, and hence have normal vector $\langle 0, 1, 0\rangle$. Thus, the plane is

$$0(x-0) + 1(y-1) + 0(z-0) = 0 \quad\Rightarrow\quad y = 1.$$

(viii) After checking all lines of the form $y = mx$ and all parabolas of the form $y = mx^2$ and always getting zero, we suspect the limit is zero. Using the squeeze theorem

$$0 \leq \frac{x^2 \sin^4 y}{3x^2 + 2y^2} \leq \frac{x^2 \sin^4 y}{3x^2} = \frac{\sin^4 y}{3} \to 0$$

as $y \to 0$. Thus, the limit is zero.

(ix) The region of integration in the xy-plane is the region enclosed by the parabola $y = 1 - x^2$ in the first quadrant. Thus, the integral represents the volume of the solid regions that lies under the plane $z = 1 - x$ and above the previously described region in the xy-plane. Your drawing should reflect this.

(x) We parameterize the surface by $\mathbf{r}(y, z) = \langle y^2 + z^2, y, z\rangle$ where $y^2 + z^2 \leq 9$. Thus,

$$A(S) = \iint_D |\mathbf{r}_y \times \mathbf{r}_z| \mathrm{d}A = \iint_D \sqrt{1 + 4(y^2 + z^2)} \mathrm{d}A = \int_0^{2\pi} \int_0^3 r\sqrt{1 + 4r^2} \mathrm{d}r \mathrm{d}\theta = \frac{\pi}{6}(37)^{3/2}.$$

(xi) We can think of each of these surfaces of level surfaces of functions: $f(x, y, z) = k$. The normal vector of the tangent plane of such a surface at a point (x_0, y_0, z_0) is $\nabla f(x_0, y_0, z_0)$. Thus, the normal vector for the tangent plane to the ellipsoid is $\mathbf{n}_1 = \langle 6, 4, 4\rangle$ and the normal vector for the tangent plane to the sphere is $\mathbf{n}_2 = -\langle 6, 4, 4\rangle$. These vectors are parallel, because they are scalar multiples of each other, and both tangent planes contain the point $(1, 1, 2)$. Since any two planes that are parallel and contain a common point must be the same plane, the ellipsoid and sphere are tangent at that point.

(xii) We can parameterize the sphere using $\mathbf{r}(\theta, \phi) = \langle \sqrt{2}\cos\theta\sin\phi, \sqrt{2}\sin\theta\sin\phi, \sqrt{2}\cos\phi\rangle$. When it intersects the cylinder, we have $2 = x^2 + y^2 + z^2 = 1 + z^2$, and so $1 = z^2 = (\sqrt{2}\cos\phi)^2$. Thus, this happens when $\phi = \pi/4$, and so the domain for the parameters defining the surface is $0 \leq \phi \leq \pi/4$, $0 \leq \theta \leq 2\pi$. As a result

$$\iint_S y^2 \mathrm{d}S = \int_0^{2\pi} \int_0^{\pi/4} 2\sin^2\theta \sin^2\phi |\mathbf{r}_\theta \times \mathbf{r}_\phi| \mathrm{d}\phi \mathrm{d}\theta = \int_0^{2\pi} \int_0^{\pi/4} 4\sin^2\theta \sin^3\phi \mathrm{d}\phi \mathrm{d} = 4\pi\left(\frac{2}{3} - \frac{5\sqrt{2}}{12}\right).$$

(xiii) We have

$$f_z = xy\cos(xyz)e^{\sin(xyz)} - \frac{2x^2yz}{(1 + z^2)^2}.$$

(xiv) Since $\mathbf{u} = \langle 1, 0, 1\rangle/\sqrt{2}$ and $\nabla f(1, 2, 3) = \langle 12, 3, 2\rangle$, we have

$$D_{\mathbf{u}}f = \langle 1, 0, 1\rangle/\sqrt{2} \cdot \langle 12, 3, 2\rangle = \frac{14}{\sqrt{2}}.$$

(xv) The relationship between the length of the hypotenuse (D), the length of the base (l) and the height (h) of the triangle is $D = \sqrt{l^2 + h^2}$. Thus

$$\mathrm{d}D = \frac{l}{\sqrt{l^2 + h^2}} \mathrm{d}l + \frac{h}{\sqrt{l^2 + h^2}} \mathrm{d}h = \frac{5}{12}(0.2) + \frac{12}{13}(0.2) = \frac{17}{65} \mathrm{m}.$$