Solutions to Quiz 1 – MA 225 B2 – Spring 2011

Question 1

(i) $\langle -5, 22, 7\rangle$

(ii) 1

(iii) $\langle 1, -2, 7\rangle$

(iv) They are not orthogonal, because their dot product is nonzero. They are not parallel, because their cross product is nonzero. Hence: neither.

Question 2 The cross product produces a vector orthogonal to both, so we compute $\langle 1, -3, 0\rangle \times \langle 0, -2, 1\rangle = \langle -3, -1, -2\rangle$. We divide by its magnitude to get a unit vector $\mathbf{u}_1 = (1/\sqrt{14})\langle -3, -1, -2\rangle$. The only other unit vector orthogonal to both is the one that points in the direction opposite to $\mathbf{u}_1$. Hence, $\mathbf{u}_2 = (1/\sqrt{14})\langle 3, 1, 2\rangle$.

Question 3

(i) The angle between the planes is the acute angle between their normal vectors. We have $\mathbf{n}_1 = \langle 3, 1, -1\rangle$ and $\mathbf{n}_2 = \langle -2, 1, 1\rangle$. If θ is the angle between them in the dot product, then

$$\cos\theta = \frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1||\mathbf{n}_2|} = -\sqrt{\frac{6}{11}}.$$

Since this is negative, $\pi/2 < \theta < \pi$, and so we need to subtract it from π to make it acute. Hence, the angle between the planes is $\pi - \cos^{-1}(-\sqrt{\frac{6}{11}})$.

(ii) The direction of the line is orthogonal to both normal vectors, so we can find it by taking the cross product of the normal vectors. Hence, $\mathbf{v} = \mathbf{n}_1 \times \mathbf{n}_2 = \langle 2, -1, 5\rangle$. To find a point on the line, we can, for example, look for the point of intersection of the planes that lies in the yz-plane, which means setting $x = 0$ in both equations for the plane. Solving the resulting equations, $y - z = 4$ and $y + z = 1$, we see that point is $(0, 5/2, -3/2)$. Thus, the equation for the line is

$$x = 2t, \qquad y = 5/2 - t, \qquad z = -3/2 + 5t.$$

(iii) A perpendicular plane with have normal vector perpendicular to the normal vectors of both given planes. Hence, the normal vector is the vector $\mathbf{v}$ that we computed in part (ii). Thus, the equation of the plane is

$$2(x - 1) - y + 5(z + 1) = 0.$$

Question 4 The $x = k$ trace is given by $z^2/9 - y^2/4 = 1 - k^2$, which is a hyperbola that opens along the z axis if $|k| < 1$ and along the y axis if $|k| > 1$. Similarly, the $z = k$ trace is given by $x^2 - y^2/4 = 1 - k^2/9$, which is a hyperbola that opens along the y axis if $|k| > 3$ and along the x axis if $|k| < 3$. The $y = k$ trace is given by $x^2 + z^2/9 = 1 + k^2/4$, which is an ellipse with the z axis as its long axis. Thus, the surface is a hyperboloid of one sheet that opens along the y axis.

Solutions to Quiz 1 – MA 225 B3 – Spring 2011

Question 1

(i) $\langle -5, 29, 7\rangle$

(ii) 2

(iii) $\langle 1, -2, 9\rangle$

(iv) They are not orthogonal, because their dot product is nonzero. They are not parallel, because their cross product is nonzero. Hence: neither.

Question 2 The cross product produces a vector orthogonal to both, so we compute $\langle 1, -3, 0\rangle \times \langle 0, -2, 1\rangle = \langle -3, -1, -2\rangle$. We divide by its magnitude to get a unit vector $\mathbf{u}_1 = (1/\sqrt{14})\langle -3, -1, -2\rangle$. The only other unit vector orthogonal to both is the one that points in the direction opposite to $\mathbf{u}_1$. Hence, $\mathbf{u}_2 = (1/\sqrt{14})\langle 3, 1, 2\rangle$.

Question 3

(i) The angle between the planes is the acute angle between their normal vectors. We have $\mathbf{n}_1 = \langle 3, 1, -1\rangle$ and $\mathbf{n}_2 = \langle -2, 1, 1\rangle$. If θ is the angle between them in the dot product, then

$$\cos\theta = \frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1||\mathbf{n}_2|} = -\sqrt{\frac{6}{11}}.$$

Since this is negative, $\pi/2 < \theta < \pi$, and so we need to subtract it from π to make it acute. Hence, the angle between the planes is $\pi - \cos^{-1}(-\sqrt{\frac{6}{11}})$.

(ii) The direction of the line is orthogonal to both normal vectors, so we can find it by taking the cross product of the normal vectors. Hence, $\mathbf{v} = \mathbf{n}_1 \times \mathbf{n}_2 = \langle 2, -1, 5\rangle$. To find a point on the line, we can, for example, look for the point of intersection of the planes that lies in the yz-plane, which means setting $x = 0$ in both equations for the plane. Solving the resulting equations, $y - z = 4$ and $y + z = 1$, we see that point is $(0, 5/2, -3/2)$. Thus, the equation for the line is

$$x = 2t, \qquad y = 5/2 - t, \qquad z = -3/2 + 5t.$$

(iii) A perpendicular plane with have normal vector perpendicular to the normal vectors of both given planes. Hence, the normal vector is the vector $\mathbf{v}$ that we computed in part (ii). Thus, the equation of the plane is

$$2(x-1) - y + 5(z+1) = 0.$$

Question 4 The $x = k$ trace is given by $z^2/9 - y^2/4 = 1 - k^2$, which is a hyperbola that opens along the z axis if $|k| < 1$ and along the y axis if $|k| > 1$. Similarly, the $z = k$ trace is given by $x^2 - y^2/4 = 1 - k^2/9$, which is a hyperbola that opens along the y axis if $|k| > 3$ and along the x axis if $|k| < 3$. The $y = k$ trace is given by $x^2 + z^2/9 = 1 + k^2/4$, which is an ellipse with the z axis as its long axis. Thus, the surface is a hyperboloid of one sheet that opens along the y axis.

Solutions to Quiz 1 – MA 225 B4 – Spring 2011

Question 1

(i) $\langle 15, -4, 7\rangle$

(ii) -1

(iii) $\langle 3, -1, -7\rangle$

(iv) They are not orthogonal, because their dot product is nonzero. They are not parallel, because their cross product is nonzero. Hence: neither.

Question 2 The cross product produces a vector orthogonal to both, so we compute $\langle 2,1,0\rangle \times \langle 0,3,1\rangle = \langle 1,-2,6\rangle$. We divide by its magnitude to get a unit vector $\mathbf{u}_1 = (1/\sqrt{41})\langle 1,-2,6\rangle$. The only other unit vector orthogonal to both is the one that points in the direction opposite to $\mathbf{u}_1$. Hence, $\mathbf{u}_2 = (1/\sqrt{41})\langle -1,2,-6\rangle$.

Question 3

(i) The angle between the planes is the acute angle between their normal vectors. We have $\mathbf{n}_1 = \langle 1,3,-1\rangle$ and $\mathbf{n}_2 = \langle 1,-2,1\rangle$. If θ is the angle between them in the dot product, then

$$\cos\theta = \frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1||\mathbf{n}_2|} = -\sqrt{\frac{6}{11}}.$$

Since this is negative, $\pi/2 < \theta < \pi$, and so we need to subtract it from π to make it acute. Hence, the angle between the planes is $\pi - \cos^{-1}(-\sqrt{\frac{6}{11}})$.

(ii) The direction of the line is orthogonal to both normal vectors, so we can find it by taking the cross product of the normal vectors. Hence, $\mathbf{v} = \mathbf{n}_1 \times \mathbf{n}_2 = \langle 1,-2,-5\rangle$. To find a point on the line, we can, for example, look for the point of intersection of the planes that lies in the yz-plane, which means setting $x = 0$ in both equations for the plane. Solving the resulting equations, $3y - z = 4$ and $-2y + z = 1$, we see that point is $(0,5,11)$. Thus, $\mathbf{r}_0 = \langle 0,5,11\rangle$, and the vector equation for the line is $\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$.

(iii) Let $\mathbf{w}$ be a vector between the given point P and any point on the line. For example, using the point found above, $\mathbf{w} = \langle 2,-4,-12\rangle$. The normal vector for the plane is then $\mathbf{n}_3 = \mathbf{v} \times \mathbf{w} = \langle 4,2,0\rangle$, where $\mathbf{v}$ is the direction of the line found above. Thus, the equation of the plane is

$$4x + 2(y-5) = 0.$$

Question 4 The $z = k$ trace is given by $x^2/9 - y^2/4 = k$, which is a hyperbola that opens along the x axis if $k > 0$ and along the y axis if $k < 0$. The $y = k$ trace is given by $z = x^2/9 - k^2/4$, which is a parabola that opens up along the z axis and gets shifted down as $|k|$ increases. The $x = k$ trace is given by $z = -y^2/4 + k^2/9$, which is a parabola that opens down along the z axis and gets shifted up as $|k|$ increases. Thus, the surface is a hyperbolic paraboloid.

Solutions to Quiz 1 – MA 225 B5 – Spring 2011

Question 1

(i) $\langle 15, -4, 28\rangle$

(ii) -1

(iii) $\langle 12, -4, -7\rangle$

(iv) They are not orthogonal, because their dot product is nonzero. They are not parallel, because their cross product is nonzero. Hence: neither.

Question 2 The cross product produces a vector orthogonal to both, so we compute $\langle -2, 1, 0\rangle \times \langle 0, 1, 1\rangle = \langle 1, 2, -2\rangle$. We divide by its magnitude to get a unit vector $\mathbf{u}_1 = (1/3)\langle 1, 2, -2\rangle$. The only other unit vector orthogonal to both is the one that points in the direction opposite to $\mathbf{u}_1$. Hence, $\mathbf{u}_2 = (1/3)\langle -1, -2, 2\rangle$.

Question 3

(i) The angle between the planes is the acute angle between their normal vectors. We have $\mathbf{n}_1 = \langle 1, 3, -2\rangle$ and $\mathbf{n}_2 = \langle 1, -1, 2\rangle$. If θ is the angle between them in the dot product, then

$$\cos\theta = \frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1||\mathbf{n}_2|} = -\sqrt{\frac{6}{14}}.$$

Since this is negative, $\pi/2 < \theta < \pi$, and so we need to subtract it from π to make it acute. Hence, the angle between the planes is $\pi - \cos^{-1}(-\sqrt{\frac{6}{14}})$.

(ii) The direction of the line is orthogonal to both normal vectors, so we can find it by taking the cross product of the normal vectors. Hence, $\mathbf{v} = \mathbf{n}_1 \times \mathbf{n}_2 = \langle 4, -4, -4\rangle$. To find a point on the line, we can, for example, look for the point of intersection of the planes that lies in the yz-plane, which means setting $x = 0$ in both equations for the plane. Solving the resulting equations, $3y - 2z = 4$ and $-y + 2z = 1$, we see that point is $(0, 5/2, 7/4)$. Thus, $\mathbf{r}_0 = \langle 0, 5/2, 7/4\rangle$, and the vector equation for the line is $\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$.

(iii) Let $\mathbf{w}$ be a vector between the given point P and any point on the line. For example, using the point found above, $\mathbf{w} = \langle 1, -5/2, -15/4\rangle$. The normal vector for the plane is then $\mathbf{n}_3 = \mathbf{v} \times \mathbf{w} = \langle 5, 11, -6\rangle$, where $\mathbf{v}$ is the direction of the line found above. Thus, the equation of the plane is

$$5(x - 1) + 11y - 6(z + 2) = 0.$$

Question 4 The $z = k$ trace is given by $x^2/9 - y^2/4 = k$, which is a hyperbola that opens along the x axis if $k > 0$ and along the y axis if $k < 0$. The $y = k$ trace is given by $z = x^2/9 - k^2/4$, which is a parabola that opens up along the z axis and gets shifted down as $|k|$ increases. The $x = k$ trace is given by $z = -y^2/4 + k^2/9$, which is a parabola that opens down along the z axis and gets shifted up as $|k|$ increases. Thus, the surface is a hyperbolic paraboloid.