Solutions to Quiz 3 – MA 225 B2 and B3 – Spring 2011

Question 1 A parameterization of the line is

$$\mathbf{r}(t) = (1-t)\langle 0,2,3\rangle + t\langle 1,6,4\rangle = \langle t, 2+4t, 3+t\rangle, \qquad 0 \le t \le 1.$$

Thus,

$$\int_C f(x,y,z)\mathrm{d}s = \int_0^1 f(\mathbf{r}(t))|\mathbf{r}'(t)|\mathrm{d}t = \sqrt{18}\int_0^1 t^2(2+4t)(3+t)\mathrm{d}t = 63\sqrt{18}/10.$$

Question 2 This region is described by $-1 \le z \le 1$, $-4 \le y \le -x^2$, and $-2 \le x \le 2$. Therefore,

$$V = \int_{-1}^{1}\int_{-2}^{2}\int_{-4}^{-x^2} 1\mathrm{d}y\mathrm{d}x\mathrm{d}z.$$

Question 3 If we try to evaluate this integral using the formula $\int_C \mathbf{F}\cdot\mathrm{d}\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t)\mathrm{d}t$, we get something difficult to evaluate, so instead we use Green's Theorem. Hence,

$$\int_C \mathbf{F}\cdot\mathrm{d}\mathbf{r} = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathrm{d}A = \iint_R \left(\frac{2}{1+(y/x)^2}\frac{-y}{x^2} - 1 + \frac{2y}{x^2+y^2}\right)\mathrm{d}A = -\iint_R 1\mathrm{d}A,$$

which is just -1 times the area of R. Since R is just a circle of radius 2, this area, and the value of the integral, is -4π.

Question 4 The integral is independent of path if the vector field is conservative, ie if there is a function f such that $\mathbf{F} = \nabla f$. This is true if $P_y = Q_x$, where $F = \langle P, Q\rangle$. Since $P_y = e^y$ and $Q_x = e^y$, the vector field is conservative and the integral is path-independent.

Question 5 If we define $u = x+y$ and $v = y-x$, then $2 \le v \le 4$, $0 \le u \le 2$ and $|\partial(x,y)/\partial(u,v)| = 1/2$. Therefore,

$$\iint_R \frac{x+y}{x-y}\mathrm{d}A = \int_2^4\int_0^2 -\frac{u}{v}\frac{1}{2}\mathrm{d}u\mathrm{d}v = -\frac{1}{2}(2)(\ln 4 - \ln 2) = -(2\ln 2 - \ln 2) = -\ln 2.$$

Solutions to Quiz 3 – MA 225 B4 and B5 – Spring 2011

Question 1 A parameterization of the line is

$$\mathbf{r}(t) = (1-t)\langle -1, 5, 0\rangle + t\langle 1, 6, 4\rangle = \langle -1+2t, 5+t, 4t\rangle, \qquad 0 \le t \le 1.$$

Thus,

$$\int_C f(x,y,z)\mathrm{d}s = \int_0^1 f(\mathbf{r}(t))|\mathbf{r}'(t)|\mathrm{d}t = 16\sqrt{21}\int_0^1 t^2(-1+2t)(5+t)\mathrm{d}t = 236\sqrt{21}/15.$$

Question 2 This region is described by $0 \le z \le 4$, $x^2 \le y \le 9$, and $-3 \le x \le 3$. Therefore,

$$V = \int_0^4 \int_{-3}^3 \int_{x^2}^9 1\mathrm{d}y\mathrm{d}x\mathrm{d}z.$$

Question 3 If we try to evaluate this integral using the formula $\int_C \mathbf{F}\cdot\mathrm{d}\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t)\mathrm{d}t$, we get something difficult to evaluate, so instead we use Green's Theorem. Hence,

$$\int_C \mathbf{F}\cdot\mathrm{d}\mathbf{r} = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathrm{d}A = \iint_R \left(\frac{2}{1+(y/x)^2}\frac{-y}{x^2} - 1 + \frac{2y}{x^2+y^2}\right)\mathrm{d}A = \iint_R 1\mathrm{d}A,$$

which is just -1 times the area of R. Since R is just a circle of radius 3, this area, and the value of the integral, is -9π.

Question 4 The integral is independent of path if the vector field is conservative, ie if there is a function f such that $\mathbf{F} = \nabla f$. This is true if $P_y = Q_x$, where $F = \langle P, Q\rangle$. Since $P_y = -e^{-x}$ and $Q_x = -e^{-x}$, the vector field is conservative and the integral is path-independent.

Question 5 If we define $u = x+y$ and $v = y-x$, then $0 \le v \le 2$, $2 \le u \le 4$ and $|\partial(x,y)/\partial(u,v)| = 1/2$. Therefore,

$$\iint_R \frac{x-y}{x+y}\mathrm{d}A = \int_0^2 \int_2^4 -\frac{v}{u}\frac{1}{2}\mathrm{d}u\mathrm{d}v = -\frac{1}{2}(2)(\ln 4 - \ln 2) = -(2\ln 2 - \ln 2) = -\ln 2.$$