Homework Assignment 2, Due Wednesday, February 5

1) (Ex 2.19, Chicone) Find a fundamental matrix solution of

$$\dot{x} = \begin{pmatrix} 1 & -\frac{1}{t} \\ 1+t & -1 \end{pmatrix} x, \quad t > 0.$$

(Hint: one solution is $x(t) = (1, t)$.)

2) Consider the space $\mathcal{L}(\mathbb{C}^n)$ with the topology generated by the operator norm,

$$\|A\| = \sup_{|v|=1} |Av|.$$

(a) Prove that the set of matrices with n distinct eigenvalues is open.

(b) Let B be an $m \times m$ Jordan block, which implies $B = \lambda I + N$ where N is nilpotent of order m. Find a matrix Q such that $Q^{-1}BQ = \lambda I + \epsilon N$. Explain why this implies that, for any matrix A, any 1's in the upper diagonal of its Jordan Canonical Form can be made arbitrarily small. Does this imply that there exists another matrix $\tilde{A} \in \mathcal{L}(\mathbb{C}^n)$ arbitrarily close to A whose eigenvectors are independent?

3) (Ex 2.36, Chicone) (Laplace transform) Recall that the definition of the Laplace transform of a (perhaps matrix valued) function f is

$$L[f](s) = \int_0^\infty e^{-s\tau} f(\tau) \mathrm{d}\tau.$$

(a) Prove that if A is an $n \times n$ matrix then

$$e^{tA} - I = \int_0^t A e^{\tau A} \mathrm{d}\tau.$$

(b) Prove that if all eigenvalues of A have negative real parts, then

$$-A^{-1} = \int_0^\infty e^{\tau A} \mathrm{d}\tau.$$

(c) Prove that if $s \in \mathbb{R}$ is sufficiently large, then

$$(sI - A)^{-1} = \int_0^\infty e^{-s\tau} e^{\tau A} \mathrm{d}\tau;$$

that is, the Laplace transform of e^{tA} is $(sI - A)^{-1}$.

(d) Solve the initial value problem $\dot{x} = Ax$, $x(0) = x_0$, using the method of Laplace transform: start by taking the Laplace transform of both sides of the equation, solve the resulting equation, then take the inverse Laplace transform.

4) Solve the initial value problem

$$\frac{dx}{dt} = Ax, \qquad x(0) = x_0, \qquad A = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$$

by using the Jordan Canconical Form of A and the matrix exponential e^{At}.

5) (Ex 2.42, Chicone)

(a) Suppose A is an $n \times n$ matrix such that $A^2 = I$. Find an explicit formula for e^{tA}.

(b) Do the same if $A^2 = -I$.

(c) Solve

$$\dot{x} = \begin{pmatrix} 2 & -5 & 8 & -12 \\ 1 & -2 & 4 & -8 \\ 0 & 0 & 2 & -5 \\ 0 & 0 & 1 & -2 \end{pmatrix} x, \qquad x(0) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

6) (Picard Iteration) Another method for proving existence of solutions to the initial value problem

$$\frac{dx}{dt} = f(x,t), \qquad x(0) = x^0$$

is to use what's known as a Picard iteration. This means defining a sequence of functions $\{x_n(t)\}_{n=0}^{\infty}$ as

$$x_0(t) = x^0, \qquad x_n(t) = x^0 + \int_0^t f(x_{n-1}(s), s)\mathrm{d}s$$

and showing that the sequence converges to the desired solution. Compute the first few functions in this sequence for the ODE $x' = Ax$. What happens in the limit as $n \to \infty$?

7) Consider the ODE

$$x' = Ax, \qquad A = \begin{pmatrix} -1 & \frac{1}{\epsilon} \\ 0 & -2 \end{pmatrix}, \qquad 0 < \epsilon \ll 1.$$

Find a solution such that $|x(0)| = 1$ but $|x(t)| \geq C/\epsilon$ when $0 < t_1 \leq t \leq t_2$, for some $0 < t_1 < t_2$. This demonstrates that, even if solutions are all decaying to zero, there can be periods of large transient growth.

8) (Ex 2.50, Chicone) Find a matrix function $t \to A(t)$ such that

$$t \to \exp\left(\int_0^t A(s)\mathrm{d}s\right)$$

is not a matrix solution of the system $\dot{x} = A(t)x$. Show that the above formula is a solution in the scalar case. When is it a solution for the matrix case?