Homework Assignment 3, Due Wednesday, February 19

1) (Ex 2.85, Chicone) Suppose that $a : \mathbb{R} \to \mathbb{R}$ is T periodic. Find the characteristic multiplier, a Floquet exponent, and the Floquet normal form for solutions to $\dot{x} = a(t)x$.

2) (Ex 2.93, Chicone) Find the principle fundamental matrix solution and Floquet normal form for the following example, which we discussed in class:

$$\dot{x} = A(t)x, \quad A(t) = \begin{pmatrix} -1 + \frac{3}{2}\cos^2 t & 1 - \frac{3}{2}\sin t \cos t \\ -1 - \frac{3}{2}\sin t \cos t & -1 + \frac{3}{2}\sin^2 t \end{pmatrix}.$$

It may help to recall that one solution is given by $x(t) = e^{t/2}(-\cos t, \sin t)$.

3) Consider Hill's equation:

$$x'' + a(t)x = \lambda x, \qquad a(t+T) = a(t),$$

where a is smooth and $\lambda \in \mathbb{R}$. Let $\Phi(t;\lambda)$ be the principal fundamental matrix solution corresponding to the associated first order system. Prove that, if $|\mathrm{Tr}(\Phi(T;\lambda))| \leq 2$, then there exists a T-periodic solution of Hill's equation. Furthermore, consider the scalar $\mathrm{Tr}(\Phi(T;\lambda))$ as a function of the real parameter λ. Indicate why, in general, one might expect bands, or intervals, $I \subset \mathbb{R}$ that correspond to values of λ for which periodic solutions exist.

4) Consider the matrix

$$A(t) = \begin{cases} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} & \text{if} \quad t < 0 \\ \begin{pmatrix} -1 & 0 \\ \alpha & 1-\alpha \end{pmatrix} & \text{if} \quad t \geq 0 \end{cases}$$

Determine the values of α for which the system

$$\frac{dx}{dt} = A(t)x + \begin{pmatrix} e^{-|t|} \\ 2e^{-|t|} \end{pmatrix}$$

has a unique solution $x(t)$ that satisfies $x \in L^2(\mathbb{R})$. Explicitly construct the unique solution for those values of α. Hint: don't focus on the discontinuity of A. Instead, use exponential dichotomies on $\mathbb{R}^+$ and $\mathbb{R}^-$. It may help to note that, given a dichotomy on $\mathbb{R}^-$, $\Phi_-^{s,u}$, the solutions to the equation $x' = A(t)x + f(t)$ that remain bounded as $t \to -\infty$ are given by

$$x_-(t) = \Phi_-^u(t,t_0)v_- + \int_{-\infty}^t \Phi_-^u(t,r)f(r)dr + \int_0^t \Phi_+^s(t,r)f(r)dr.$$

5) Consider

$$\dot{x} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} x.$$

(a) Write down explicitly the exponential dichotomy for this system, based upon the discussion in class for constant-coefficient equations. (This is a dichotomy on all of $\mathbb{R}$, so also on both $\mathbb{R}^+$ and $\mathbb{R}^-$.)

(b) Using the general definition of an exponential dichotomy and the projection

$$P(r) = \begin{pmatrix} 0 & 0 \\ e^{-2r} & 1 \end{pmatrix},$$

to construct another dichotomy that is defined on $J = \mathbb{R}^+$. This shows that dichotomies are not necessarily unique!

(c) Compute the null space and range for the projection operator (or family of projection operators) associated with each of these dichotomies. Are they the same or different? If they are the same, is it necessary that any dichotomy for this equation on $\mathbb{R}^+$ would share this property? If they are different, why are they allowed to differ?