Homework Assignment 4, Due Monday, March 3

1) Consider the ODE

$$\begin{aligned} y' &= -y \\ z' &= z + y^2 \end{aligned}$$

and the mapping $H(y,z) = (y, z + (y^2/3))$. Prove that H is a conjugacy (as in the Hartman-Grobman Theorem) between the flow corresponding to the above nonlinear system and its associated linearized flow.

2) Consider an ODE of the form

$$\dot{x} = Sx + F(x,y), \quad \dot{y} = Cy + G(x,y), \quad (x,y) \in \mathbb{R}^k \times \mathbb{R}^l \tag{0.1}$$

where all the eigenvalues of S have negative real part, all the eigenvalues of C have zero real part, and F, G satisfy the hypotheses of the invariant manifold theorem we discussed in class. Thus, the "unstable" invariant manifold guaranteed by the theorem is really a center manifold, given by

$$W^c(0,0) = \{(x,y) : x = \alpha(y)\}.$$

(a) Show that the function $\alpha(y)$ must satisfy the differential equation

$$S\alpha + F(\alpha, y) = D\alpha[Cy + G(\alpha, y)] \quad \forall y \in \mathbb{R}^l.$$

(b) Given an initial condition (x_0, y_0), let $(x(t,x_0), y(t,y_0))$ denote the corresponding solution to (0.1). Prove that there exist constants $C, \gamma > 0$ such that

$$|x(t,x_0) - \alpha(y(t,y_0))| \leq Ce^{-\gamma t}|x_0 - \alpha(y_0)| \quad \forall t \geq 0.$$

(c) Explain in words why the dynamics of (0.1) are qualitatively determined by the equation

$$\dot{y} = Cy + G(\alpha(y), y). \tag{0.2}$$

Prove that, if $y = 0$ is an unstable fixed point of (0.2), then the origin is an unstable fixed point of (0.1). (Note: a fixed point y_0 is said to be stable if, for any $\epsilon > 0$, there exists a $\delta > 0$ such that if $|y_0| < \delta$ then $|y(t,y_0)| < \epsilon$ for all $t \geq 0$. A fixed point is unstable if it is not stable.)

It is also possible to prove that, if $y = 0$ is a stable fixed point of (0.2), then the origin is a stable fixed point of (0.1). Sometimes this is encapsulated in the statement: "The nonlinear stability of the fixed point at the origin is determined by the dynamics on the center manifold."

3) (Chicone Ex 4.3) This problem essentially asks you to apply the result of the previous exercise in a particular context. This technique is referred to as a "center manifold reduction."

(a) For the following system, find an explicit formula for the center manifold at the origin:

$$\begin{aligned} \dot{x} &= -x + y^2 - 2x^2 \\ \dot{y} &= \epsilon y - xy. \end{aligned}$$

(Hint: you know the center manifold is given by a function of the form $x = \alpha(y)$. Try assuming the function α has a Taylor series expansion. Use invariance of the center manifold to determine the coefficients in the expansion.)

(b) Determine the dynamics on the center manifold. Is the origin stable? How does its stability depend on ϵ? Sketch the (x, y) phase plane near the origin for different values of ϵ, including $\epsilon = 0$.

4) Consider the system

$$\begin{aligned} x' &= y \\ y' &= -y + ax^2 + bxy. \end{aligned}$$

Assume that the center manifold is analytic in a neighborhood of the origin, and thus has a Taylor series expansion. Compute the first two terms in this expansion and use them determine the stability of the origin, which will depend on the values of a and b. (Note: it might help to change basis via the Jordan Normal Form so that the eigendirections are on the coordinate axes, before computing the terms in the expansion.)

5) Consider the system

$$\begin{aligned} x' &= -x^3 \\ y' &= -y + x^2. \end{aligned}$$

Compute an expansion of the center manifold, as above, using an Ansatz of the form $y = h(x) = \sum_{k=2}^{\infty} a_k x^k$, and derive a recursion relation for the coefficients a_k. Conclude that the center manifold is not analytic at the origin. Verify this by solving for the integral curves to find an explicit formula for the center manifold (which is C^∞ but not analytic).