Homework Assignment 5, Due Monday, March 24

1) Consider the ODE $\dot{u} = f(u)$ with periodic orbit Γ. In class I said that, given any $p \in \Gamma$ and Σ' transverse to Γ at p, there exists an open set $\Sigma \subset \Sigma'$ and a first return map

$$T : \Sigma \to \mathbb{R}, \qquad u(T(\sigma), \sigma) \in \Sigma' \quad \text{if} \quad \sigma \in \Sigma.$$

Furthermore, if f is C^1, then so is T. Use the implicit function theorem to justify that statement, both the existence of T and the fact that it is C^1 (at least if f is smooth).

2) (Chicone 2.134/137) Consider the following ODE in polar coordinates.

$$\dot{r} = r(1 - r), \quad \dot{\theta} = r.$$

Pick a Poincaré section and compute its return time map and the associated Poincaré map. Determine the eigenvalues of the linearization of this map at the point in the periodic orbit. Also, for each p in the periodic orbit, compute the sets Γ_p that have the same asymptotic phase as the solution $u(t, p)$.

3) Suppose that the ODE $\dot{x} = f(x)$ with $x \in \mathbb{R}^2$ has a limit cycle $\gamma(t)$ with period T. Consider

$$\lambda = \int_0^T (\nabla \cdot f)(\gamma(t)) \mathrm{d}t.$$

Prove that, if $\lambda < 0$, then γ is asymptotically stable, while if $\lambda > 0$ then γ is asymptotically unstable.