Homework Assignment 6, Due Wednesday, April 9

1) (Chicone , Excercise 1.73) Show that the system

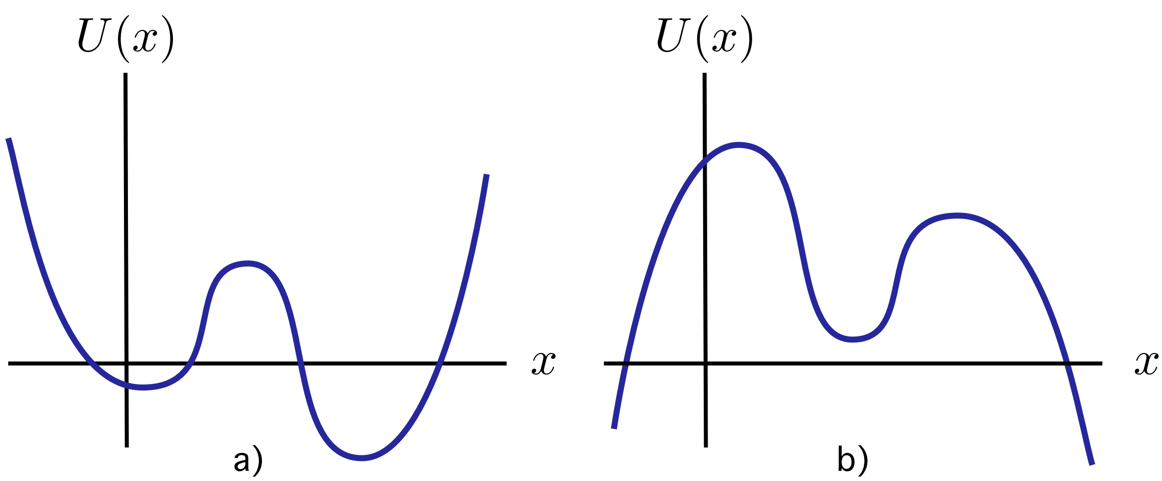
$$\begin{aligned} \dot{I}_1 &= I_1 \cos(\theta_1 - \theta_2) \\ \dot{I}_2 &= -I_1 \cos(\theta_1 - \theta_2) \\ \dot{\theta}_1 &= -1 - \sin(\theta_1 - \theta_2) \\ \dot{\theta}_2 &= 1 \end{aligned}$$

is Hamiltonian. Also, find a first integral F for the system that is independent of the Hamiltonian H.

2) Consider the Hamiltonian system defined by

$$H(x,y) = \frac{1}{2}y^2 + U(x), \quad x, y \in \mathbb{R},$$

where $U(x)$ has the form given in the following pictures. For each picture, draw qualitatively the complete phase plane for the system.



3) Prove that Hamiltonian ODEs are area preserving. In other words, if $\mathcal{R}_0 := \{x \in R\}$ for some smooth region $R \subset \mathbb{R}^n$, prove that

$$\int_{\mathcal{R}_0} \mathrm{d}x = \int_{\mathcal{R}_t} \mathrm{d}x \qquad \forall t \in \mathbb{R},$$

where $\mathcal{R}_t := \{u(t;x) : x \in R\}$ and $u(t;x)$ is the solution to a Hamiltonian ODE with initial condition x. (You can assume the solutions are defined for all time.)

4) (Chicone Ex. 1.76) Prove that a gradient system cannot have a periodic orbit.

5) (Chicone Ex 1.201) Prove that the system

$$\dot{x} = x - y - x^3, \quad \dot{y} = x + y - y^3, \quad x, y \in \mathbb{R},$$

has a unique periodic orbit that is globally attracting on $\mathbb{R}^2 \setminus \{0\}$. (This means that, for any $\xi \in \mathbb{R}^2$, $\xi \neq 0$, $\phi(t;\xi)$ converges to the periodic orbit.)