Homework Assignment 7, Due Wednesday, April 23

1) Consider the two systems

$$\dot{x} = Ax, \quad \dot{y} = By, \quad x, y \in \mathbb{R}^n,$$

where both A and B are constant $n \times n$ matrices whose eigenvalues all have negative real part. Prove that the flows generated by the two equations are topologically conjugate using the following steps.

(a) Recall that there are two adapted norms, $|\cdot|_A$ and $|\cdot|_B$, and two positive constants a, b such that $|e^{At}v|_A \le e^{-at}|v|_A$ and $|e^{Bt}v|_B \le e^{-bt}|v|_B$ for all $t \ge 0$. Show that this implies that $|e^{At}v|_A \ge e^{-at}|v|_A$ and $|e^{Bt}v|_B \ge e^{-bt}|v|_B$ for all $t \le 0$.

(b) Define the spheres $S_A = \{x \in \mathbb{R}^n : |x|_A = 1\}$ and $S_B = \{x \in \mathbb{R}^n : |x|_B = 1\}$. Consider the function defined by $h_0(x) = x/|x|_B$. Show h_0 is a homeomorphism mapping S_A to S_B.

(c) Show that for any $x \neq 0$ there exists a unique time $\tau(x) \in \mathbb{R}$, depending continuously on x, such that $|e^{A\tau(x)}x|_A = 1$. Also show that $\tau(e^{At}x) = \tau(x) - t$.

(d) Define a map $h : \mathbb{R}^n \to \mathbb{R}^n$ via

$$h(x) = \begin{cases} e^{-B\tau(x)} h_0(e^{A\tau(x)}x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

Prove that this defines a topological conjugacy between the flows e^{At} and e^{Bt}.

Finally,

(e) Explain how to extend this argument to prove that any two constant coefficient linear ODEs such that the matrices are hyperbolic and have the same number of eigenvalues with negative and positive real parts are conjugate. (You don't have to prove this in detail, but write a few sentences explaining how it could be done.)

2) Recall the saddle-node and pitchfork bifurcations can be described by

$$\dot{x} = \alpha - x^2 =: F(x, \alpha), \qquad \dot{y} = \beta y - y^3 =: G(y, \beta).$$

Consider the two-dimensional surfaces defined by the functions F and G.

(a) Compute the tangent planes for these surfaces at each equilibrium point of each equation.

(b) Show that F is not tangent to the (x, α) plane at its bifurcation value, but that G is tangent to the (y, β) plane at its bifurcation value.

(c) Consider the following perturbation of the pitchfork bifurcation:

$$\dot{y} = \beta y - y^3 + \epsilon y^2.$$

Fix $\epsilon > 0$ and compute the bifurcation diagram as β is varied.

3) Consider the ODE

$$\begin{aligned} \dot{x} &= \alpha - x^2 + xy \\ \dot{y} &= -2y + x^2 + y^2. \end{aligned}$$

(a) Use the theorem we proved using Lyapunov-Schmidt reduction to show that there is a saddle node bifurcation at $(x, y, \alpha) = (0, 0, 0)$.

(b) Using a Taylor series expansion, determine an approximate formula for the center manifold (up to and including all quadratic terms), and an equation for the reduced dynamics on the center manifold.

4) Prove that the Van der Pol system undergoes a Hopf bifurcation for an appropriate value of ϵ.

$$\ddot{x} + (x^2 - \epsilon)\dot{x} + x = 0.$$