SOLUTIONS – Test 1A – MA 225 – Spring 2015

February 13, 2015

Question 1 [20 points] Let

$$\mathbf{u} = \langle 2, -1, 3\rangle, \qquad \mathbf{v} = \langle 4, 5, 0\rangle, \qquad \mathbf{w} = \langle 1, -1, -4\rangle$$

be vectors in $\mathbb{R}^3$. For each of the following, if the quantity makes sense, compute it. If it does not make sense, explain why.

(i) $(2\mathbf{u}) \cdot \mathbf{v}$

$$(2\mathbf{u}) \cdot \mathbf{v} = \langle 4, -2, 6\rangle \cdot \langle 4, 5, 0\rangle = 4(4) - 2(5) + 6(0) = 6$$

(ii) $(\mathbf{u} - \mathbf{v}) \times \mathbf{w}$

$$\begin{aligned} (\mathbf{u} - \mathbf{v}) \times \mathbf{w} &= \langle -2, -6, 3\rangle \times \langle 1, -1, -4\rangle \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & -6 & 3 \\ 1 & -1 & -4 \end{vmatrix} \\ &= (24 + 3)\mathbf{i} - (8 - 3)\mathbf{j} + (2 + 6)\mathbf{k} = \langle 27, -5, 8\rangle. \end{aligned}$$

(iii) $\mathbf{u} \times (\mathbf{v} \cdot \mathbf{w})$

This doesn't make sense because $\mathbf{v} \cdot \mathbf{w}$ is a scalar, not a vector, and the cross product is an operation between two vectors.

(iv) $|\mathbf{w}|$

$$|\mathbf{w}| = \sqrt{1^2 + (-1)^2 + (-4)^2} = \sqrt{18}.$$

Question 2 [20 points] Let

$$\mathbf{u} = \langle 1, -5, 4\rangle, \qquad \mathbf{v} = \langle -2, c, -8\rangle, \qquad \mathbf{w} = \langle 3, 1, 0\rangle$$

be vectors in $\mathbb{R}^3$, where c is a scalar.

(i) Find two unit vectors that are parallel to $\mathbf{u}$.

$$\frac{\mathbf{u}}{|\mathbf{u}|} = \frac{1}{\sqrt{1 + 25 + 16}}\langle 1, -5, 4\rangle = \frac{1}{\sqrt{42}}\langle 1, -5, 4\rangle, \qquad -\frac{\mathbf{u}}{|\mathbf{u}|} = -\frac{1}{\sqrt{42}}\langle 1, -5, 4\rangle.$$

(ii) Find a value of c such that $\mathbf{u}$ and $\mathbf{v}$ are parallel.

By definition, parallel vectors are scalar multiples of each other. If we take $c = 10$, then

$$\mathbf{v} = \langle -2, 10, -8\rangle = -2\langle 1, -5, 4\rangle = -2\mathbf{u}.$$

(iii) Find a value of c such that $\mathbf{u}$ and $\mathbf{v}$ are orthogonal.
Orthogonal vectors have zero dot product, and so we need

$$0 = \mathbf{u} \cdot \mathbf{v} = -2 - 5c - 32 = -34 - 5c.$$

Therefore, we must take $c = -34/5$.

(iv) Compute $\mathrm{proj}_{\mathbf{u}}\mathbf{w}$.

$$\mathrm{proj}_{\mathbf{u}}\mathbf{w} = \frac{\mathbf{u} \cdot \mathbf{w}}{\mathbf{u} \cdot \mathbf{u}}\mathbf{u} = \frac{(3-5+0)}{42}\langle 1, -5, 4\rangle = -\frac{1}{21}\langle 1, -5, 4\rangle.$$

Question 3 [18 points]

(i) Give a geometric description of the set of points (x, y, z) that satisfy

$$x^2 + y^2 + z^2 - 4x + 6z \geq 10.$$

First, notice that

$$x^2 - 4x = (x-2)^2 - 4, \qquad z^2 + 6z = (z+3)^2 - 9,$$

and so we have

$$(x-2)^2 - 4 + y^2 + (z+3)^2 - 9 \geq 10 \qquad \Rightarrow \qquad (x-2)^2 + y^2 + (z+3)^2 \geq 23.$$

This is the set of points that lie on or outside the sphere of radius $\sqrt{23}$ with center $(2, 0, -3)$.

(ii) Write down an equation describing the plane that is parallel to the xz-plane and that contains the point $(-3, 2, -8)$.
In this plane, the x and z components can be anything, but y must be 2. Hence, the equation is

$$y = 2.$$

(iii) Describe the set of all vectors whose projection onto the unit coordinate vector $\mathbf{k}$ is zero, and draw a picture of the collection of all such vectors.
This is the set of all vectors $\mathbf{u} = \langle u_1, u_2, u_3\rangle$ such that

$$\mathrm{proj}_{\mathbf{k}}\mathbf{u} = (\mathbf{k} \cdot \mathbf{u})\mathbf{k} = 0.$$

Therefore, $\mathbf{k}$ and $\mathbf{u}$ must be orthogonal, which means $0 = \mathbf{k} \cdot \mathbf{u} = u_3$. This is any vector of the form $\mathbf{u} = \langle u_1, u_2, 0\rangle$, which is any vector that lies in the xy-plane. (I'm not including a sketch of the xy-plane in these solutions, but it should have been in your solution on the test.)

Question 4 [20 points]

(i) Find an equation of the line containing the points $(0, 2, 3)$ and $(1, -4, 2)$.

For the direction vector, one can take the vector going from the first to the second point, which is $\langle 1, -6, -1\rangle$. If one takes the first point as the point on the line, one obtains

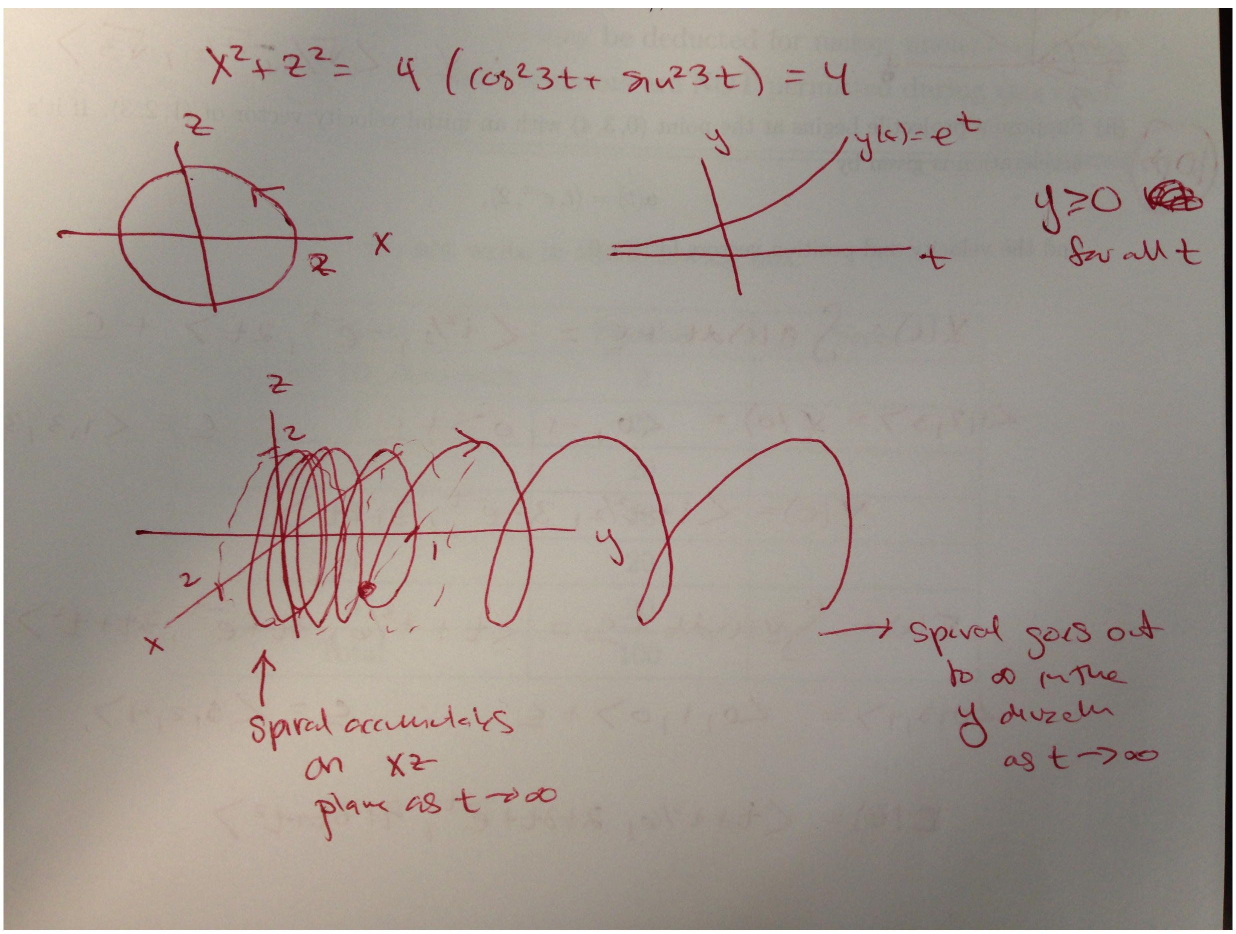
$$\mathbf{r}(t) = \langle 0, 2, 3\rangle + t\langle 1, -6, -1\rangle, \qquad -\infty < t < \infty$$

(The above is not the only correct solution to this problem.)

(ii) Sketch the curve described by the following function and describe in words all key aspects of your picture.

$$\mathbf{r}(t) = \langle 2\cos 3t, e^t, 2\sin 3t\rangle, \qquad -\infty < t < \infty$$

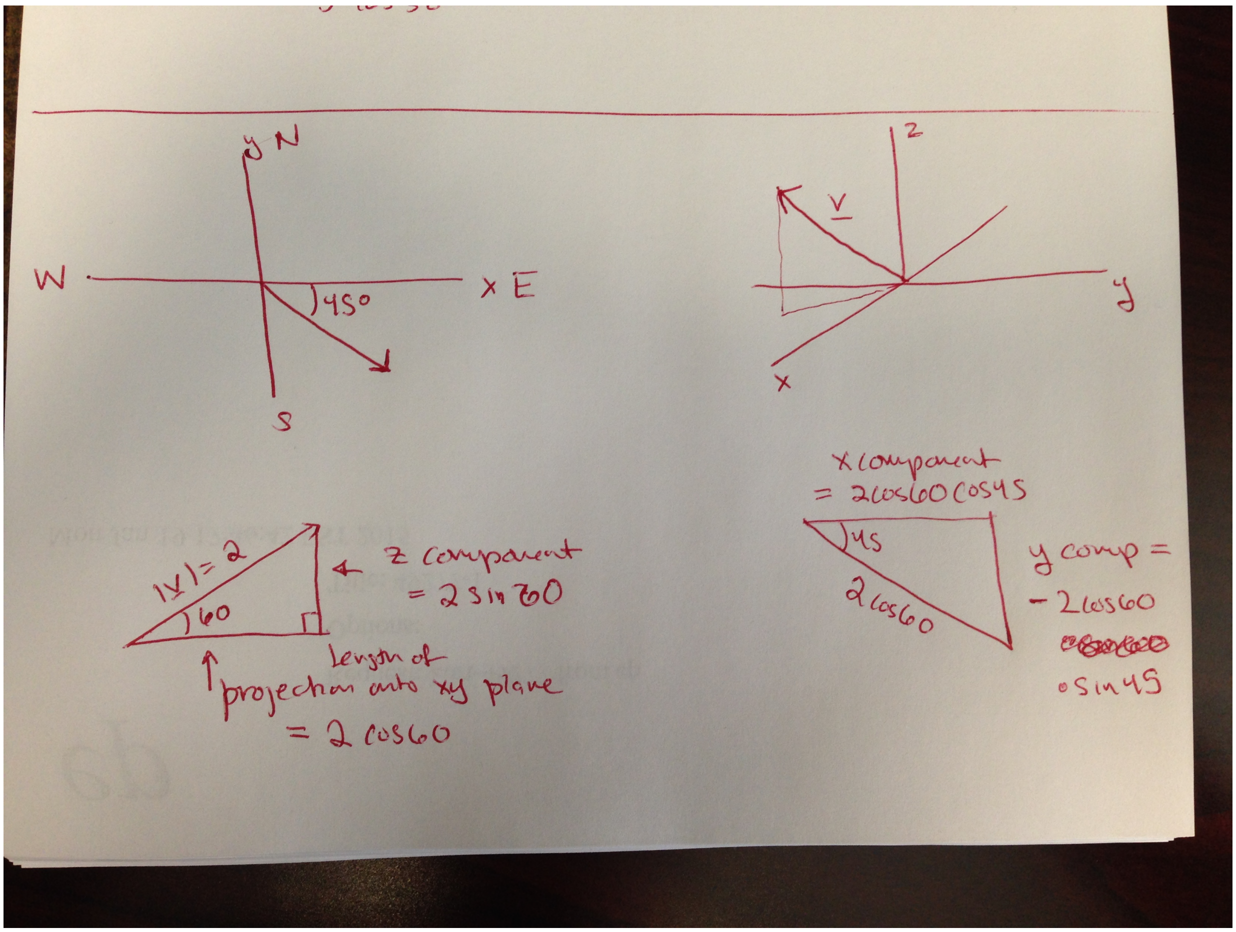
First, notice that $x^2 + z^2 = 4(\cos^2 3t + \sin^2 3t) = 4$, and so when projected onto the xz-plane this curve just rotates around the circle of radius 2 counterclockwise. Furthermore, since $y(t) = e^t$, we see that $y \geq 0$, y approaches zero as $t \to -\infty$, and t approaches infinity as $t \to \infty$. Putting this information together, we find



(There is a small typo in the above figure - it should say "spiral accumulates on the xz-plane as $t \to -\infty$.")

Question 5 [20 points]

(i) A fish in the water is climbing at an angle of 60 degrees above the horizontal with a heading to the southeast. If its speed is 2 mi/hr, find the three components of its velocity vector.
As described in the below figure, we can compute the z-component of the velocity to be $2\sin 60 = \sqrt{3}$. This implies that the x- and y-components are $2\cos 60\cos 45 = 2(1/2)(\sqrt{2}/2) = \sqrt{2}/2$ and $-2\cos 60\sin 45 = -\sqrt{2}/2$, respectively. Hence, $\mathbf{v} = \langle\sqrt{2}/2, -\sqrt{2}/2, \sqrt{3}\rangle$.



(ii) Suppose a projectile begins at the point $(0, 3, 4)$ with an initial velocity vector of $\langle 1, 2, 3\rangle$. If its acceleration is given by

$$\mathbf{a}(t) = \langle t, e^{-t}, 2\rangle,$$

find the velocity and position vectors for $t \geq 0$.

$$\mathbf{v}(t) = \int \mathbf{a}(t)\mathrm{d}t + \mathbf{c} = \langle\frac{t^2}{2}, -e^{-t}, 2t\rangle + \mathbf{c}.$$

Since, $\langle 1, 2, 3\rangle = \mathbf{v}(0) = \langle 0, -1, 0\rangle + \mathbf{c}$, we need the constant vector to be $\langle 1, 3, 3\rangle$. As a result,

$$\mathbf{v}(t) = \langle 1 + \frac{t^2}{2}, 3 - e^{-t}, 3 + 2t\rangle.$$

To compute the position,

$$\mathbf{r}(t) = \int \mathbf{v}(t)\mathrm{d}t + \mathbf{c} = \langle t + \frac{1}{6}t^3, 3t + e^{-t}, 3t + t^2 \rangle + \mathbf{c}.$$

Also, $\langle 0, 3, 4 \rangle = \mathbf{r}(0) = \langle 0, 1, 0 \rangle + \mathbf{c}$, and so $\mathbf{c} = \langle 0, 2, 4 \rangle$. Hence,

$$\mathbf{r}(t) = \langle t + \frac{1}{6}t^3, 2 + 3t + e^{-t}, 4 + 3t + t^2 \rangle.$$