SOLUTIONS – Test 1B – MA 225 – Spring 2015

February 13, 2015

Question 1 [20 points] Let

$$\mathbf{u} = \langle 3, 2, -1\rangle, \qquad \mathbf{v} = \langle 0, 2, 1\rangle, \qquad \mathbf{w} = \langle 5, -1, 1\rangle$$

be vectors in $\mathbb{R}^3$. For each of the following, if the quantity makes sense, compute it. If it does not make sense, explain why.

(i) $\mathbf{u} \cdot (5\mathbf{v})$

$$\mathbf{u} \cdot (5\mathbf{v}) = \langle 3, 2, -1\rangle \cdot \langle 0, 10, 5\rangle = 3(0) + 2(10) - 1(5) = 15.$$

(ii) $\mathbf{u} \times (\mathbf{v} - \mathbf{w})$

$$\begin{aligned} \mathbf{u} \times (\mathbf{v} - \mathbf{w}) &= \langle 3, 2, -1\rangle \times \langle -5, 3, 0\rangle \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 2 & -1 \\ -5 & 3 & 0 \end{vmatrix} \\ &= (0+3)\mathbf{i} - (0-5)\mathbf{j} + (9+10)\mathbf{k} = \langle 3, 5, 19\rangle. \end{aligned}$$

(iii) $\mathbf{u} \cdot (\mathbf{v} \cdot \mathbf{w})$

This doesn't make sense because $\mathbf{v} \cdot \mathbf{w}$ is a scalar, not a vector, and the dot product is an operation between two vectors.

(iv) $|\mathbf{u}|$

$$|\mathbf{u}| = \sqrt{3^2 + 2^2 + (-1)^2} = \sqrt{14}.$$

Question 2 [20 points] Let

$$\mathbf{u} = \langle 2, -3, 1\rangle, \qquad \mathbf{v} = \langle c, 6, -2\rangle, \qquad \mathbf{w} = \langle 0, 2, 5\rangle$$

be vectors in $\mathbb{R}^3$, where c is a scalar.

(i) Compute $\text{proj}_{\mathbf{u}}\mathbf{w}$.

$$\text{proj}_{\mathbf{u}}\mathbf{w} = \frac{\mathbf{u} \cdot \mathbf{w}}{\mathbf{u} \cdot \mathbf{u}}\mathbf{u} = \frac{(0-6+5)}{(4+9+1)}\langle 2, -3, 1\rangle = -\frac{1}{14}\langle 2, -3, 1\rangle.$$

(ii) Find a value of c such that $\mathbf{u}$ and $\mathbf{v}$ are parallel.

By definition, parallel vectors are scalar multiples of each other. If we take $c = -4$, then

$$\mathbf{v} = \langle -4, 6, -2\rangle = -2\langle 2, -3, 1\rangle = -2\mathbf{u}.$$

(iii) Find a value of c such that $\mathbf{u}$ and $\mathbf{v}$ are orthogonal.
Orthogonal vectors have zero dot product, and so we need

$$0 = \mathbf{u} \cdot \mathbf{v} = 2c - 18 - 2 = 2c - 20.$$

Therefore, we must take $c = 10$.

(iv) Find two unit vectors that are parallel to $\mathbf{u}$.

$$\frac{\mathbf{u}}{|\mathbf{u}|} = \frac{1}{\sqrt{4+9+1}}\langle 2, -3, 1\rangle = \frac{1}{\sqrt{14}}\langle 2, -3, 1\rangle, \qquad -\frac{\mathbf{u}}{|\mathbf{u}|} = -\frac{1}{\sqrt{14}}\langle 2, -3, 1\rangle.$$

Question 3 [18 points]

(i) Describe the set of all vectors whose projection onto the unit coordinate vector $\mathbf{j}$ is zero, and draw a picture of the collection of all such vectors.
This is the set of all vectors $\mathbf{u} = \langle u_1, u_2, u_3\rangle$ such that

$$\mathrm{proj}_{\mathbf{j}}\mathbf{u} = (\mathbf{j} \cdot \mathbf{u})\mathbf{j} = 0.$$

Therefore, $\mathbf{j}$ and $\mathbf{u}$ must be orthogonal, which means $0 = \mathbf{j} \cdot \mathbf{u} = u_2$. This is any vector of the form $\mathbf{u} = \langle u_1, 0, u_3\rangle$, which is any vector that lies in the xz-plane. (I'm not including a sketch of the xz-plane in these solutions, but it should have been a part of your solution for the test.)

(ii) Write down an equation describing the plane that is parallel to the yz-plane and that contains the point $(2, 1, -7)$.
In this plane, the y and z components can be anything, but x must be 2. Hence, the equation is

$$x = 2.$$

(iii) Give a geometric description of the set of points (x, y, z) that satisfy

$$x^2 + y^2 + z^2 + 8y - 2z \geq 4.$$

First, notice that

$$y^2 + 8y = (y+4)^2 - 16, \qquad z^2 - 2z = (z-1)^2 - 1,$$

and so we have

$$x^2 + (y+4)^2 - 16 + (z-1)^2 - 1 \geq 4 \qquad \Rightarrow \qquad x^2 + (y+4)^2 + (z-1)^2 \geq 21.$$

This is the set of points that lie on or outside the sphere or radius $\sqrt{21}$ with center $(0, -4, 1)$.

Question 4 [20 points]

(i) Find an equation of the line containing the points $(1, -1, 2)$ and $(3, 1, 4)$.
For the direction vector, one can take the vector going from the first to the second point, which is $\langle 2, 2, 2\rangle$. If one takes the first point as the point on the line, one obtains

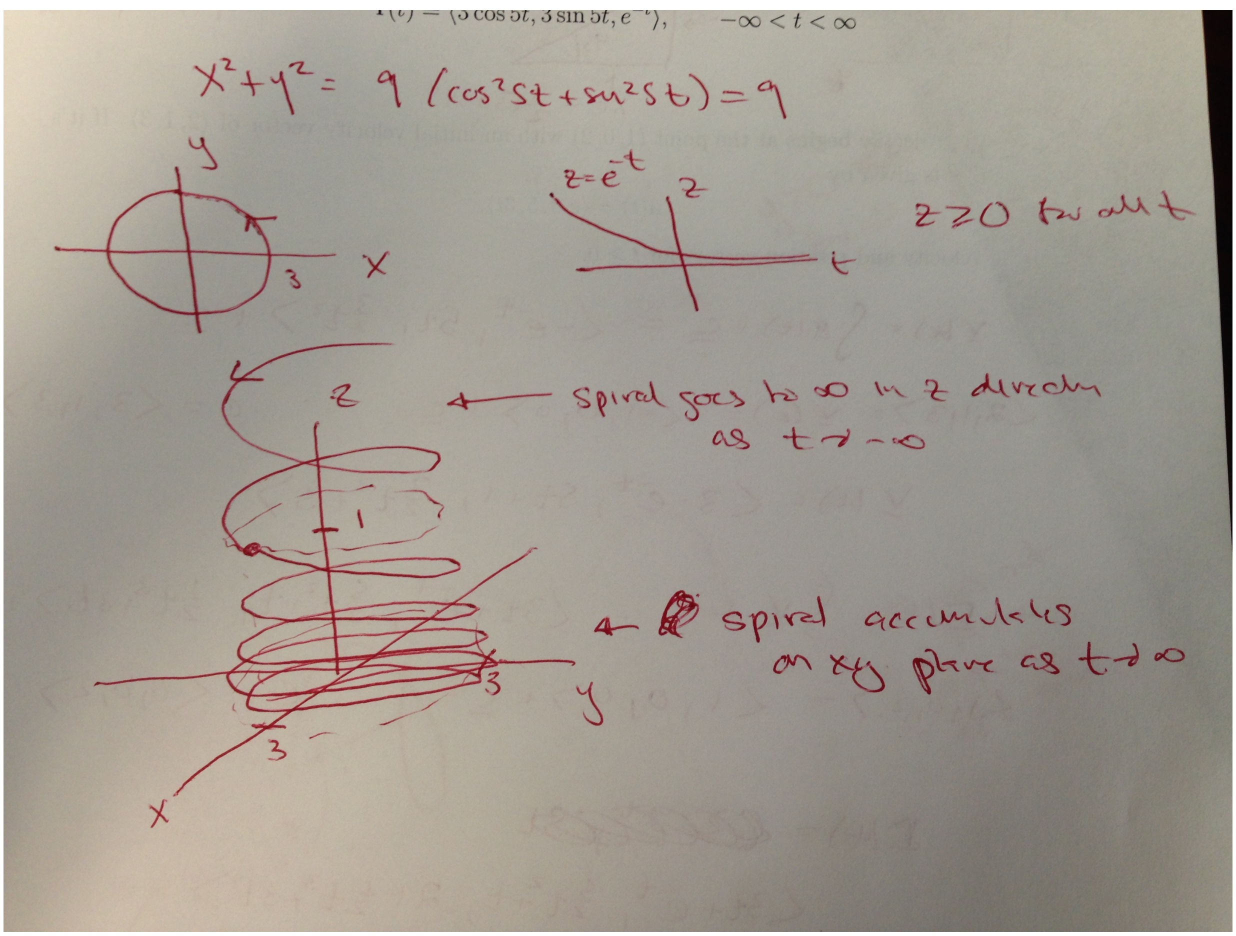
$$\mathbf{r}(t) = \langle 1, -1, 2\rangle + t\langle 2, 2, 2\rangle, \qquad -\infty < t < \infty$$

(There is more than one correct answer to the above question.)

(ii) Sketch the curve described by the following function and describe in words all key aspects of your picture.

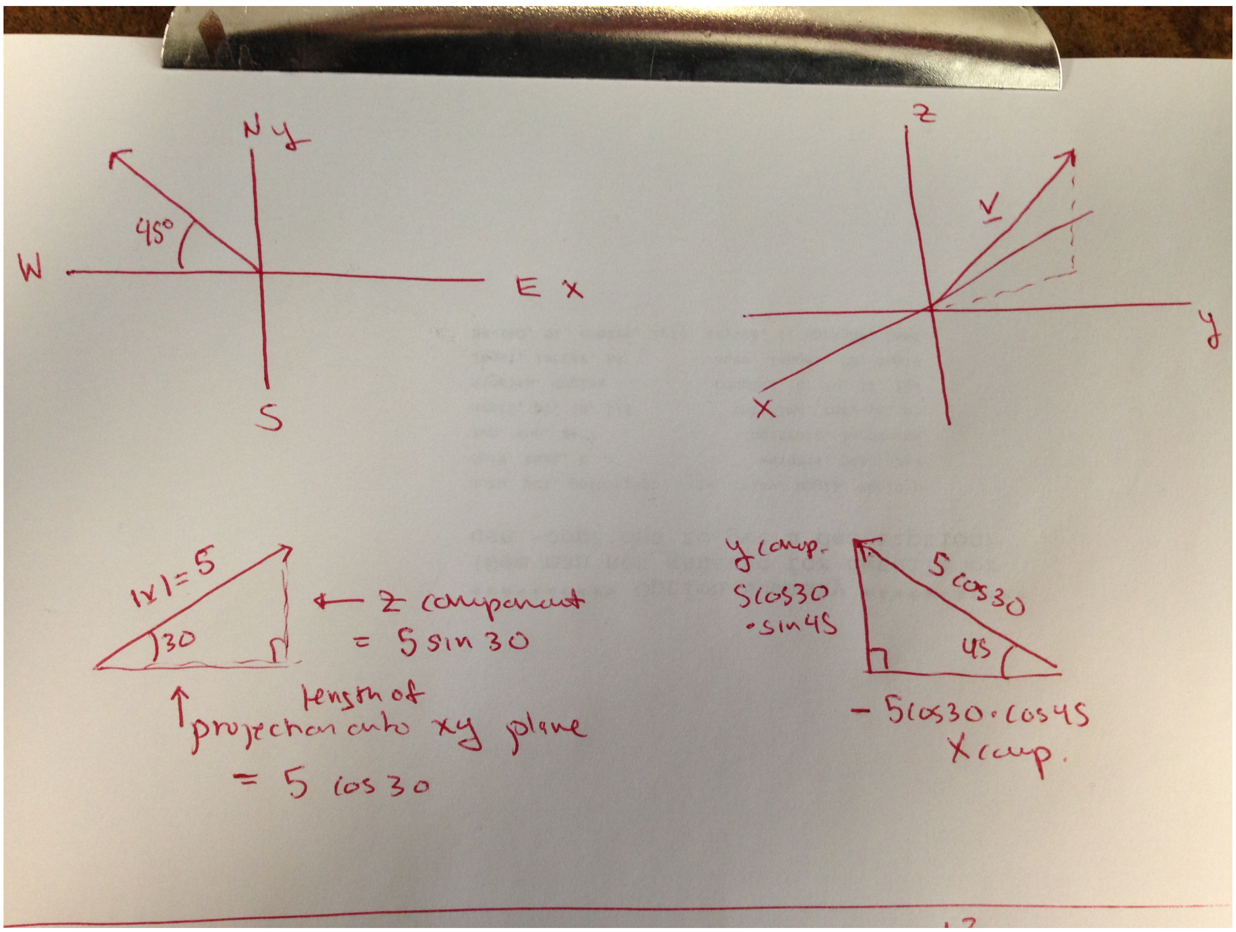
$$\mathbf{r}(t) = \langle 3\cos 5t, 3\sin 5t, e^{-t}\rangle, \qquad -\infty < t < \infty$$

First, notice that $x^2 + y^2 = 9(\cos^2 5t + \sin^2 5t) = 9$, and so when projected onto the xy-plane this curve just rotates around the circle of radius 3 counterclockwise. Furthermore, since $z(t) = e^{-t}$, we see that $z \geq 0$, z approaches zero as $t \to \infty$, and z approaches infinity as $t \to -\infty$. Putting this information together, we find



Question 5 [20 points]

(i) A dolphin in the water is climbing at an angle of 30 degrees above the horizontal with a heading to the northwest. If its speed is 5 mi/hr, find the three components of its velocity vector.
As described in the below figure, we can compute the z-component of the velocity to be $5 \sin 30 = 5/2$. This implies that the x- and y-components are $-5 \cos 30 \cos 45 = -5(\sqrt{3}/2)(\sqrt{2}/2) = -5\sqrt{6}/4$ and $5 \cos 30 \sin 45 = 5\sqrt{6}/4$, respectively. Hence, $\mathbf{v} = \langle -5\sqrt{6}/4, 5\sqrt{6}/4, 5/2 \rangle$.



(ii) Suppose a projectile begins at the point $(1, 0, 2)$ with an initial velocity vector of $\langle 2, 1, 3 \rangle$. If its acceleration is given by

$$\mathbf{a}(t) = \langle e^{-t}, 5, 3t \rangle,$$

find the velocity and position vectors for $t \geq 0$.

$$\mathbf{v}(t) = \int \mathbf{a}(t) \mathrm{d}t + \mathbf{c} = \langle -e^{-t}, 5t, \frac{3}{2}t^2 \rangle + \mathbf{c}.$$

Since, $\langle 2, 1, 3 \rangle = \mathbf{v}(0) = \langle -1, 0, 0 \rangle + \mathbf{c}$, we need the constant vector to be $\langle 3, 1, 3 \rangle$. As a result,

$$\mathbf{v}(t) = \langle 3 - e^{-t}, 1 + 5t, 3 + \frac{3}{2}t^2 \rangle.$$

To compute the position,

$$\mathbf{r}(t) = \int \mathbf{v}(t)\mathrm{d}t + \mathbf{c} = \langle 3t + e^{-t}, t + \frac{5}{2}t^2, 3t + \frac{1}{2}t^3 \rangle + \mathbf{c}.$$

Also, $\langle 1, 0, 2 \rangle = \mathbf{r}(0) = \langle 1, 0, 0 \rangle + \mathbf{c}$, and so $\mathbf{c} = \langle 0, 0, 2 \rangle$. Hence,

$$\mathbf{r}(t) = \langle 3t + e^{-t}, t + \frac{5}{2}t^2, 2 + 3t + \frac{1}{2}t^3 \rangle.$$