SOLUTIONS – Test 2B – MA 225 – Spring 2015

March 25, 2015

Question 1 [24 points]

(i) **[8 points]** Compute the directional derivative of the following function at the given point in the direction of the given vector.

$$h(x, y) = 4 + 3x^2 - 2y^2, \qquad P(2, 5), \qquad \langle 3, -1\rangle.$$

The directional derivative is given by $D_{\mathbf{u}}h = \nabla h \cdot \mathbf{u}$, where $\mathbf{u}$ is a unit vector in the direction of the given vector. We find

$$\nabla h(2, 5) = \langle 6x, -4y\rangle|_{(2,5)} = \langle 12, -20\rangle, \qquad \mathbf{u} = \frac{1}{\sqrt{10}}\langle 3, -1\rangle,$$

and so $D_{\mathbf{u}}h(1, 4) = \langle 12, -20\rangle \cdot (1/\sqrt{10})\langle 3, -1\rangle = 56/\sqrt{10}$.

(ii) **[8 points]** For the above function $h(x, y)$ in part (ii), find a vector that points in a direction of no change in the function at $P(2, 3)$.

The function doesn't change in any direction that is orthogonal its gradient. Therefore, we need a vector that is orthogonal to $\nabla h(2, 5) = \langle 12, -20\rangle = 4\langle 3, -5\rangle$. Such a vector is given by $\langle 5, 3\rangle$ (or any scalar multiple of this vector).

(iii) **[8 points]** Let $g(x, y) = xe^{y^2}$, $x(t) = \cos t$, and $y(t) = 3t^2$. Compute dg/dt. Make sure your answer is in terms of t only.

By the chain rule,

$$\frac{dg}{dt} = \frac{\partial g}{\partial x}\frac{dx}{dt} + \frac{\partial g}{\partial y}\frac{dy}{dt}.$$

We can compute $g_x = e^{y^2} = e^{9t^4}$, $g_y = 2xye^{y^2} = 6t^2 \cos te^{9t^4}$, $x'(t) = -\sin t$, and $y'(t) = 6t$. Hence,

$$\frac{dg}{dt} = -\sin te^{9t^4} + 36t^3 \cos te^{9t^4}.$$

Question 2 [20 points]

(i) **[8 points]** Find the equation of the plane that contains the following point and line.

$$(3, -2, 1), \qquad \mathbf{r}_1(t) = \langle 2, 3, 4\rangle + t\langle 0, 5, -1\rangle, \quad -\infty < t < \infty$$

To find a normal vector of the plane, we can use two vectors that lie in the plane and take their cross product. One is the direction vector of the given line, and we can choose another to be the vector $\langle -1, 5, 3\rangle$, which is the vector between the points $(3, -2, 1)$ and $(2, 3, 4)$, which both lie on the plane. Hence,

$$\mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 5 & -1 \\ -1 & 5 & 3 \end{vmatrix} = \langle 20, 1, 5\rangle,$$

and the plane is given by $20(x - 3) + (y + 2) + 5(z - 1) = 0$. (There is more than one correct answer. For example, one could have used the point $(2, 3, 4)$ as the point on the plane, instead of $(3, -2, 1)$.)

(ii) **[12 points]** Find the point on the plane $x + 2y - z = 5$ nearest the point $P(2, 1, 2)$. (Do not use Lagrange multipliers to solve this problem. Please use another method.)

We need to minimize the distance function

$$d(x, y, z) = (x - 2)^2 + (y - 1)^2 + (z - 2)^2.$$

(The distance between a point (x, y, z) and the given point is the square root of the above formula, but it is equivalent to minimize the above function, without the square root, which makes the calculation easier.) To make this a function of two variables only, we use the constraint $z = x + 2y - 5$, and so we need to minimize the function

$$f(x, y) = (x - 2)^2 + (y - 1)^2 + (x + 2y - 7)^2.$$

To find the critical points, we must solve

$$0 = f_x = 2(x-2)+2(x+2y-7) = 4x+4y-18, \qquad 0 = f_y = 2(y-1)+4(x+2y-7) = 4x+10y-30.$$

We find the solution to be $x = 5/2$ and $y = 2$, and so the closest point is $(5/2, 2, 3/2)$. One could check, using the second derivative test, that this is actually a minimum. However, thinking about the geometry of a point relative to a plane, we can see this must be the minimum.

Question 3 [20 points]

(i) **[8 points]** Determine the domain and range of the following function. Please be sure to justify your answer.

$$f(x, y) = 8e^{-\sqrt{9-x^2-y^2}}$$

The domain of this function is the set $\{(x, y) : 9 - x^2 - y^2 \geq 0\} = \{(x, y) : x^2 + y^2 \leq 9\}$, which is the circle of radius three with center the origin and its interior. Since $e^{-3} \leq e^{-\sqrt{9-x^2-y^2}} \leq 1$, by properties of the exponential function, the range is the interval $[8e^{-3}, 8]$.

(ii) **[12 points]** Sketch the level curves of the following function.

$$f(x, y) = 4\ln(x - 3y^2 + 2)$$

The level curves are given by the set of points (x, y) that satisfy

$$z_0 = 4\ln(x - 3y^2 + 2) \qquad \Rightarrow \qquad x = 3y^2 + e^{\frac{z_0}{4}} - 2.$$

This is a family of parabolas in the xy-plane that open along the positive x-axis and intersect the x-axis at the point $e^{\frac{z_0}{4}} - 2$. However, it is not all parabolas of the form $x = 3y^2 + c$ for some constant c, because $-\infty < z_0 < \infty$, which implies that $-2 < e^{\frac{z_0}{4}} - 2 < \infty$. Hence, the level curves are all such parabolas with $c > -2$. (Your answer should include a sketch of these level curves, in addition to the above explanation.)

Question 4 [18 points]

(i) [**8 points**] True or false: if

$$\lim_{(x,0)\to(0,0)} f(x,0) = L, \qquad \lim_{(0,y)\to(0,0)} f(0,y) = L,$$

then $\lim_{(x,y)\to(0,0)} f(x,y)$ necessarily exists and is equal to L. Be sure to justify your answer.

This is false. The two given limits concern only two ways in which (x, y) can approach $(0, 0)$: along the x and y axis, respectively. Therefore, it is possible that if (x, y) approaches the origin along a different path, say one of the form $y = mx$, that one might find a different limiting values, which would imply that the limit does not exist. (An explicit example where the above conditions hold with $L = 0$ but the limit does not exist is $f(x, y) = xy/(x^2 + y^2)$.)

(ii) [**10 points**] Evaluate the following limit or determine that it does not exist.

$$\lim_{(x,y)\to(1,0)} \frac{x \sin y}{y(x+1)}$$

This function is well behaved as $x \to 1$, so we really only need to worry about what's happening in y. By L'Hopital's rule,

$$\lim_{y\to 0} \frac{\sin y}{y} = \lim_{y\to 0} \frac{\cos y}{1} = 1.$$

Hence,

$$\lim_{(x,y)\to(1,0)} \frac{x \sin y}{y(x+1)} = \frac{1}{1+1} \lim_{y\to 0} \frac{\sin y}{y} = \frac{1}{2}.$$

Question 5 [16 points] Consider the surface described by the equation

$$4z^2 - \frac{x^2}{9} + y^2 - 1 = 0$$

Sketch the traces in the three coordinate planes. Then sketch the complete surface in $\mathbb{R}^3$. Please be sure to justify your answer.

The xy-trace is given by the equation $-x^2/9 + y^2 = 1$. This is a hyperbola in the xy-plane that intersects the y axis at the points $(x, y) = (0, \pm 1)$ and does not intersect the x axis. (Your answer should include a sketch of this hyperbola.) The yz-trace is given by the equation $4z^2 + y^2 = 1$. This is an ellipse in the yz-plane that intersects the z axis at the points $(y, z) = (0, \pm 1/2)$ and intersects the y axis at the points $(\pm 1, 0)$. (Your answer should include a sketch of this ellipse.) The xz-trace is given by the equation $4z^2 - x^2/9 = 1$. This is a hyperbola in the xz-plane that intersects the z axis at the points $(x, z) = (0, \pm 1/2)$ and does not intersect the x axis. (Your answer should include a sketch of this hyperbola.) Putting this information together, we find a hyperboloid of one sheet that is oriented along the x-axis. (You did not necessarily need to know this was called a hyperboloid of one sheet, but your answer should have included a sketch very similar to that of Example 10 in Chapter 12.1, but oriented along the x axis, rather than the z axis.)