Test 3A – MA 225 – Spring 2015 - Solutions

Question 1 [20 points] Evaluate the following two integrals using a method of your choice.

(i) **[10 points]**

$$\begin{aligned}\int_0^2 \int_0^{4-y} (x+y)\mathrm{d}x\mathrm{d}y &= \int_0^2 \left(\frac{x^2}{2} + xy\right)\Big|_{x=0}^{x=4-y}\mathrm{d}x \\ &= \int_0^2 \left(8 - \frac{1}{2}y^2\right)\mathrm{d}y = 8y - \frac{y^3}{6}\Big|_0^2 = 16 - \frac{4}{3}\end{aligned}$$

(ii) **[10 points]** If C is the line segment from $(2,2)$ to $(8,8)$, it can be parameterized by

$$\mathbf{r}(t) = \langle 2,2\rangle + t\langle 1,1\rangle, \qquad 0 \le t \le 6.$$

(Note there is more than one correct way to do this.) Therefore, $|\mathbf{r}'(t)| = |\langle 1,1\rangle| = \sqrt{2}$, and

$$\begin{aligned}\int_C (x^2+y^2)\ \mathrm{d}s &= \int_0^6 [(2+t)^2 + (2+t)^2]\sqrt{2}\mathrm{d}t \\ &= 2\sqrt{2}\int_0^6 (2+t)^2\mathrm{d}t = \frac{2\sqrt{2}}{3}(2+t)^3|_0^6 = \frac{2\sqrt{2}}{3}[8^3 - 2^3] = \frac{1008\sqrt{2}}{3}.\end{aligned}$$

Question 2 [18 points] Consider

$$f(x,y,z) = xy\sin z, \qquad \mathbf{F}(x,y,z) = \langle x, y^2z, x-y\rangle, \qquad C : \mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$$

(i) **[7 points]** $\nabla \cdot \mathbf{F} = 1 + 2yz + 0 = 1 + 2yz$.

(ii) **[7 points]** $\nabla \times \mathbf{F} = \langle -1-y^2, -1, 0\rangle$.

(iii) For each of the following, state if the quantity is a scalar, a vector, or undefined. (You do not need to compute the quantity.) Be sure to justify your answer.

(a) **[2 points]** $\int_C f\mathrm{d}s$ This is a scalar, because f is scalar-valued.

(b) **[2 points]** $\int_C \mathbf{F}\cdot\mathrm{d}\mathbf{r}$ This is also a scalar, because $\mathbf{F}$ and $\mathrm{d}\mathbf{r}$ are vectors, and the dot product of two vectors is a scalar.

(There were other correct explanations, for example relating the integrals to certain physical quantities, which are scalar-valued.)

Question 3 [20 points]

(i) Consider the following integral

$$\int_0^2 \int_0^{4-x^2} \frac{xe^{2y}}{4-y} \mathrm{d}y\mathrm{d}x$$

(a) **[5 points]** Sketch the region of integration.
The parabola $y = 4 - x^2$ opens down and contains the points $(0, 4)$ and $(2, 0)$. The region is below this parabola and above the x-axis, for x between 0 and 2. (Your answer should include a sketch of this.)

(b) **[7 points]** Reverse the order of integration. (Note, you do not need to evaluate the resulting integral.)

$$\int_0^4 \int_0^{\sqrt{4-y^2}} \frac{xe^{2y}}{4-y} \mathrm{d}x\mathrm{d}y$$

(ii) **[8 points]** Identify and sketch the surface given in spherical coordinates by $\{(\rho, \varphi, \theta) : \rho = 2\csc\varphi, \quad 0 < \varphi < \pi\}$.
Notice that, in spherical coordinates, $x^2 + y^2 = \rho^2 \sin^2 \varphi$. The given equation is equivalent to $\rho \sin\varphi = 2$, and squaring both sides gives

$$4 = \rho^2 \sin^2 \varphi = x^2 + y^2.$$

This is a cylinder of radius 2 with the origin as its center, that lies along the z axis.

Question 4 [20 points]

(i) **[10 points]** Set up, but do not evaluate, a double integral that represents the volume of the part of the cylinder $x^2 + y^2 = 1$ bounded above by the plane $z = 12 - x - y$ and below by $z = 0$.

$$\int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (12 - x - y)\mathrm{d}y\mathrm{d}x$$

(There is more than one correct answer.)

(ii) **[10 points]** Evaluate the integral

$$\iiint_D \frac{1}{(x^2 + y^2 + z^2)^{3/2}} \mathrm{d}V,$$

where D is the region between the spheres of radius 1 and 2, centered at the origin.
Converting to spherical coordinates, we find

$$\int_0^\pi \int_0^{2\pi} \int_1^2 \frac{1}{\rho^3} \rho^2 \sin\varphi \mathrm{d}\rho\mathrm{d}\theta\mathrm{d}\varphi = 2\pi(-\cos\varphi|_0^\pi)(\ln\rho|_1^2) = 4\pi\ln 2.$$

Question 5 [20 points]

(i) [**5 points**] Is the vector field $\mathbf{F} = \langle 2x^3 + xy^2, 2y^3 - x^2y \rangle$ conservative? Be sure to justify your answer.

$$f_y = 2xy, \qquad g_x = -2xy.$$

Since these are not equal, the vector field is not conservative.

(ii) [**5 points**] State the Fundamental Theorem for Line Integrals. (No justification needed here - just a correct statement of the theorem.)
Let $\mathbf{F}$ be a continuous vector field on a region R (in $\mathbb{R}^2$ or $\mathbb{R}^3$). There exists a potential function φ such that $\mathbf{F} = \nabla\varphi$ if and only if

$$\int_C \mathbf{F} \cdot \mathbf{T} \mathrm{d}s = \varphi(B) - \varphi(A),$$

for any smooth oriented curve C from A to B lying entirely in R. (You could have written the integral on the left hand side in one of its other forms.)

(iii) [**10 points**] Consider the line integral

$$\oint_C \mathbf{F} \cdot \mathbf{n} \; \mathrm{d}s,$$

where $\mathbf{F} = \langle 5y \sin x, x^2 + y^2 \rangle$, C is the circle of radius 1, centered at the origin, oriented counterclockwise, and $\mathbf{n}$ is the outward unit normal vector on the curve. Write down, but do not evaluate, an equivalent double integral.
We apply Green's theorem in the form

$$\oint_C \mathbf{F} \cdot \mathbf{n} \; \mathrm{d}s = \iint_R (f_x + g_y) \mathrm{d}A = \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (5y \cos x + 2y) \mathrm{d}y \mathrm{d}x,$$

where R is the region (a filled in circle) whose boundary is the curve C.