Test 3B – MA 225 – Spring 2015 - Solutions

Question 1 [20 points] Evaluate the following two integrals using a method of your choice.

(i) **[10 points]**

$$\int_0^5 \int_0^{3-x} (x+y)\mathrm{d}y\mathrm{d}x = \int_0^5 \left(xy + \frac{y^2}{2}\right)\Big|_{y=0}^{y=3-x}\mathrm{d}x$$
$$= \int_0^5 \left(\frac{9}{2} - \frac{1}{2}x^2\right)\mathrm{d}x = \frac{9}{2}x - \frac{x^3}{6}\Big|_0^5 = \frac{45}{2} - \frac{125}{6} = \frac{5}{3}.$$

(ii) **[10 points]** If C is the line segment from $(3,3)$ to $(5,5)$, it can be parameterized by

$$\mathbf{r}(t) = \langle 3,3\rangle + t\langle 1,1\rangle, \qquad 0 \le t \le 2.$$

(Note there is more than one correct way to do this.) Therefore, $|\mathbf{r}'(t)| = |\langle 1,1\rangle| = \sqrt{2}$, and

$$\int_C (4x^2 - y^2)\ \mathrm{d}s = \int_0^2 [4(3+t)^2 - (3+t)^2]\sqrt{2}\mathrm{d}t$$
$$= 3\sqrt{2}\int_0^2 (3+t)^2\mathrm{d}t = \sqrt{2}(3+t)^3|_0^2 = \sqrt{2}[125-27] = 98\sqrt{2}.$$

Question 2 [18 points] Consider

$$f(x,y,z) = yz\sin x, \qquad \mathbf{F}(x,y,z) = \langle zx, x, y - z^2\rangle, \qquad C : \mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$$

(i) **[7 points]** $\nabla \cdot \mathbf{F} = z + 0 - 2z = -z.$

(ii) **[7 points]** $\nabla \times \mathbf{F} = \langle 1, x, 1\rangle.$

(iii) For each of the following, state if the quantity is a scalar, a vector, or undefined. (You do not need to compute the quantity.) Be sure to justify your answer.

(a) **[2 points]** $\int_C x^2\ \mathrm{d}x + xy\ \mathrm{d}y$ is a scalar, because all terms in the integrand are scalar-valued.

(b) **[2 points]** $\int_C \mathbf{F}\cdot\mathbf{T}\ \mathrm{d}s$ is a scalar, because $\mathbf{F}$ and $\mathbf{T}$ are both vectors, and their dot product is a scalar.

(There were other correct explanations, for example relating the integrals to certain physical quantities, which are scalar-valued.)

Question 3 [20 points]

(i) Consider the following integral

$$\int_0^2 \int_{x^2+1}^5 \frac{ye^{2x}}{5-y} \mathrm{d}y \mathrm{d}x$$

(a) **[5 points]** Sketch the region of integration.
The parabola $y = x^2 + 1$ opens up and contains the points $(0, 1)$ and $(2, 5)$. Thus, the region of integration is in the first quadrant, above the parabola, and below the line $y = 5$. (Your answer should contain a sketch of this.)

(b) **[7 points]** Reverse the order of integration. (Note, you do not need to evaluate the resulting integral.)

$$\int_1^5 \int_0^{\sqrt{y-1}} \frac{ye^{2x}}{5-y} \mathrm{d}x \mathrm{d}y.$$

(ii) **[8 points]** Identify and sketch the surface given in spherical coordinates by $\{(\rho, \varphi, \theta) : \rho \sin \varphi = 4, \quad 0 < \varphi < \pi\}$.
Notice that, in spherical coordinates, $x^2 + y^2 = \rho^2 \sin^2 \varphi$. The given equation is equivalent to $\rho \sin \varphi = 4$, and squaring both sides gives

$$16 = \rho^2 \sin^2 \varphi = x^2 + y^2.$$

This is a cylinder of radius 4 with the origin as its center, that lies along the z axis.

Question 4 [20 points]

(i) **[10 points]** Set up, but do not evaluate, a double integral that represents the volume of the part of the cylinder $x^2 + y^2 = 4$ bounded above by the plane $z = 15 - x - y$ and below by $z = 0$. Make sure you clearly state the limits of integration and order of integration.

$$\int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (15 - x - y) \mathrm{d}y \mathrm{d}x$$

(There is more than one correct answer.)

(ii) **[10 points]** Evaluate the integral

$$\iiint_D \frac{1}{(x^2 + y^2 + z^2)^{5/2}} \mathrm{d}V,$$

where D is the region between the spheres of radius 2 and 3, centered at the origin.
Converting to spherical coordinates, we find

$$\int_0^\pi \int_0^{2\pi} \int_2^3 \frac{1}{\rho^5} \rho^2 \sin \varphi \mathrm{d}\rho \mathrm{d}\theta \mathrm{d}\varphi = 2\pi(-\cos \varphi|_0^\pi)(-\frac{1}{2\rho^2}|_2^3) = \frac{5\pi}{18}.$$

Question 5 [20 points]

(i) **[5 points]** Is the vector field $\mathbf{F} = \langle 3x^4 + 2xy^2, y^3 - 2x^2y \rangle$ conservative? Be sure to justify your answer.

$$f_y = 4xy, \qquad g_x = -4xy$$

Since these are not equal, the vector field is not conservative.

(ii) **[5 points]** State the Fundamental Theorem for Line Integrals. (No justification needed here - just a correct statement of the theorem.)
Let $\mathbf{F}$ be a continuous vector field on a region R (in $\mathbb{R}^2$ or $\mathbb{R}^3$). There exists a potential function φ such that $\mathbf{F} = \nabla\varphi$ if and only if

$$\int_C \mathbf{F} \cdot \mathbf{T} \mathrm{d}s = \varphi(B) - \varphi(A),$$

for any smooth oriented curve C from A to B lying entirely in R. (You could have written the integral on the left hand side in one of its other forms.)

(iii) **[10 points]** Consider the line integral

$$\oint_C \mathbf{F} \cdot \mathrm{d}\mathbf{r},$$

where $\mathbf{F} = \langle 5x \cos y, xy^2 \rangle$, C is the circle of radius 2, centered at the origin, oriented counterclockwise, and $\mathbf{n}$ is the outward unit normal vector on the curve. Write down, but do not evaluate, an equivalent double integral.
We apply Green's theorem in the form

$$\oint_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \iint_R (g_x - f_y) \mathrm{d}A = \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (y^2 + 5x \sin y) \mathrm{d}y \mathrm{d}x,$$

where R is the region (a filled in circle) whose boundary is the curve C.