MA 226 Differential Equations
Summer II, 2003 Exam #2

INSTRUCTIONS: Please read all problems carefully and write your solutions on the paper provided. Show all your work - no credit will be given unless all work is shown. Don't forget to put your name on your solution sheets.

1. (5 points) For the following matrix, compute the eigenvalues and find an associated eigenvector for each eigenvalue.

$$\mathbf{A} = \begin{pmatrix} 2 & 1 \\ -3 & 6 \end{pmatrix}$$

2. (8 points) Suppose that you are given the linear system $\frac{d\mathbf{Y}}{dt} = \mathbf{AY}$ with matrix

$$\mathbf{A} = \begin{pmatrix} -5 & -1 \\ 1 & -3 \end{pmatrix}$$

and told that $\lambda = -4$ is a repeated eigenvalue with associated eigenvector $\mathbf{V} = (1, -1)$. Please state the general solution of the system.

3. Consider the linear system $\frac{d\mathbf{Y}}{dt} = \mathbf{AY}$, where $\mathbf{A}$ has eigenvalue $\lambda = 1+i$ with associated eigenvector $(1+i, 1)$.

a) (7 points) Find the general solution to the system. (There should be no complex numbers in your answer.)

b) (3 points) Find the particular solution with $\mathbf{Y}(0) = (0, 2)$.

4. (10 points) Prove that $\mathbf{Y}(t) = e^{\lambda t}\mathbf{V}$ is a solution to the equation $\frac{d\mathbf{Y}}{dt} = \mathbf{AY}$, if λ is an eigenvalue of $\mathbf{A}$ with associated eigenvector $\mathbf{V}$.

5. Consider the system $\frac{d\mathbf{Y}}{dt} = \mathbf{AY}$.

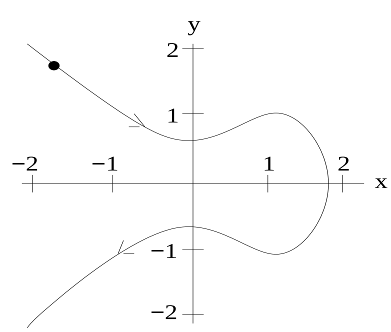
   a) (7 points) Suppose $\lambda_1 = -2$ is an eigenvalue of $\mathbf{A}$ with associated eigenvector $\mathbf{V_1} = (1, 1)$ and $\lambda_2 = -5$ is an eigenvalue of $\mathbf{A}$ with associated eigenvector $\mathbf{V_2} = (1, -3)$. Sketch the phase plane of the system.

   b) (8 points) Suppose $\lambda_1 = -1$ is an eigenvalue of $\mathbf{A}$ with associated eigenvector $\mathbf{V_1} = (1, -1)$ and $\lambda_2 = 4$ is an eigenvalue of $\mathbf{A}$ with associated eigenvector $\mathbf{V_2} = (1, 2)$. Consider the solution with initial condition $(x(0), y(0)) = (1, 1)$.

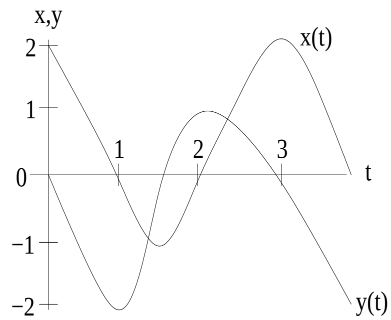
      i) Describe its behavior as t increases to $+\infty$.
      Describe its behavior as t decreases to $-\infty$

      ii) Do all solutions of the system have the same behavior?
      If not, what else can happen? Justify your answer.

6. a) (5 points) For the following curve in the xy phase plane, please draw the corresponding $x(t)$ and $y(t)$ graphs on the same set of axes. The point $(x(0), y(0))$ is shown by a dot.



b) (5 points) For the following $x(t)$ and $y(t)$ graphs, please draw the corresponding curve in the xy phase plane, indicating the point $(x(0), y(0))$ with a dot.



7. Consider the family of systems $\frac{d\mathbf{Y}}{dt} = \mathbf{AY}$ depending on the parameter α where

$$\mathbf{A} = \begin{pmatrix} \alpha & \alpha^2 - \alpha \\ 1 & \alpha \end{pmatrix}$$

a) (4 points) Sketch the corresponding curve in the trace-determinant plane.

b) (12 points) Describe the different types of behavior exhibited by the system as α increases.

c) (4 points) Identify the bifurcation values of α, where the type of the system changes.

8. Consider the harmonic oscillator we discussed in class.

a) (16 points) Please state the four classification types of the harmonic oscillator. For each type, draw a typical phase plane of the related linear system and briefly describe the corresponding motion of the mass spring system. (You don't need to write down any equations for this part.)

b) Suppose the mass is given by $m = 2$, the spring constant by $k = 1$, and the damping by $b = 3$.

i) (2 points) Write the corresponding second-order equation and first order system.

ii) (4 points) State the classification type, natural period, natural frequency, and rotation direction (if applicable).

9. **Extra Credit** This problem is worth 5 points - no partial credit will be given.

Consider the following system in polar coordinates

$$\begin{aligned} \frac{dr}{dt} &= r(1-r) \\ \frac{d\theta}{dt} &= 1 \end{aligned}$$

where $x = r\cos(\theta)$ and $y = r\sin(\theta)$. Sketch the phase portrait in the xy plane. (Hint: Very little calculation is necessary, but be sure to justify your answer.)