MA 226 Differential Equations
Summer II, 2003 Final Exam

INSTRUCTIONS: Please read all problems carefully and write your solutions on the paper provided. Show all your work - no credit will be given unless all work is shown. Don't forget to put your name on your solution sheets.

1. The following questions are each worth 4 points. All of them require little or no computation, but make sure you justify your answer.

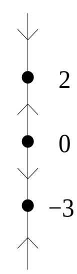
   a) Which of the following pairs of vectors are linearly independent?

   i) $\mathbf{V}_1 = (-1, 3)$, $\mathbf{V}_2 = (2, -6)$

   ii) $\mathbf{V}_1 = (2, 1)$, $\mathbf{V}_2 = (0, 1)$

   iii) $\mathbf{V}_1 = (1/2, 3/4)$, $\mathbf{V}_2 = (-4, -6)$

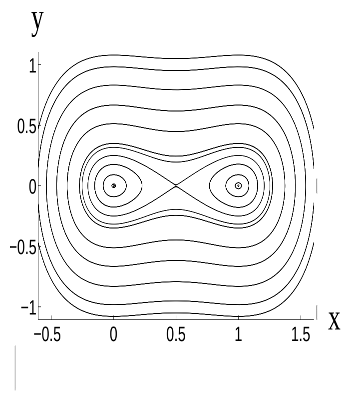
   b) Give an explicit equation for a differential equation with the following phase line:



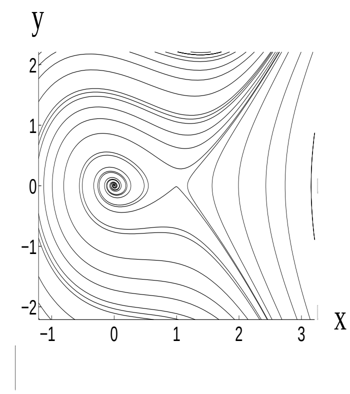
   c) Describe the behavior of solutions to a forced harmonic oscillator with mass $m = 1$, damping coefficient $b = 0$, spring constant $k = 4$, and forcing function $f(t) = \cos(2t)$. Sketch a graph illustrating this behavior.

   d) Sketch the trace determinant plane for linear systems, indicating the type of behavior the system exhibits in each region.

e) Can either of the following two phase planes be the phase plane of a Hamiltonian system? Could either be the phase plane of a Gradient system?



i)



ii)

2. Consider the following differential equation with parameter α.

$$\frac{dy}{dt} = y^2 + \alpha y + 4$$

a) (4 points) Determine the bifurcation value(s) for α.

b) (4 points) Sketch the phase line for values of α slightly large than, equal to, and slightly greater than each bifurcation value.

3. Consider the following nonlinear system:

$$\frac{dx}{dt} = xy^2 - x$$

$$\frac{dy}{dt} = x - y$$

You may restrict your attention to the first quadrant ($x \geq 0$, $y \geq 0$).

a) (3 points) Determine all of the equilibrium points.

b) (4 points) Using the Jacobian, classify each equilibrium point. Then sketch the phase portrait of the system near each equilibrium point.

c) (4 points) Find the x- and y-nullclines and draw arrows on each nullcline indicating the behavior of the vector field.

d) (4 points) Draw the complete phase plane of the system.

4. (8 points) Solve the following differential equation using the method of Laplace transforms.

$$\frac{dy}{dt} = -3y + u_4(t)e^{-2(t-4)}, \quad y(0) = 1$$

where

$$u_4(t) = \begin{cases} 0 & \text{if } t < 4, \\ 1 & \text{if } t \geq 4. \end{cases}$$

5. Given the function $K(x, y) = \frac{2}{1+x^2+y^2}$

a) (3 points) Sketch the level sets of $K(x, y)$.

b) (3 points) State the system of differential equations that has $K(x, y)$ as its conserved quantity, and sketch the phase plane for the system.

c) (3 points) State the gradient system associated to the function $K(x, y)$ and sketch the phase plane for the system.

6. (6 points) Is the following system a Hamiltonian system? If so construct its associated Hamiltonian function.

$$\frac{dx}{dt} = 2x - e^y$$

$$\frac{dy}{dt} = -2x - 2y$$

7. Consider the following initial value problem:

$$\frac{dy}{dt} = \frac{1}{y(1-t)}; \quad y(0) = 1$$

a) (3 points) Solve the initial value problem.

b) (3 points) State the domain of definition of the solution.

c) (4 points) Describe what happens to the solution as it approaches the limits of its domain of definition. The existence and uniqueness theorem guarantees that a solution to this initial value problem must exist, but there are values of t for which this solution does not exist. Why isn't this a contradiction?

8. Find the general solution of the following differential equations.

a) (5 points)

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 4y = \cos(3t)$$

Please use the technique of complexification we discussed in class to solve this problem.

b) (5 points)

$$\frac{dx}{dt} = xy$$

$$\frac{dy}{dt} = -2y + t$$

c) (5 points)

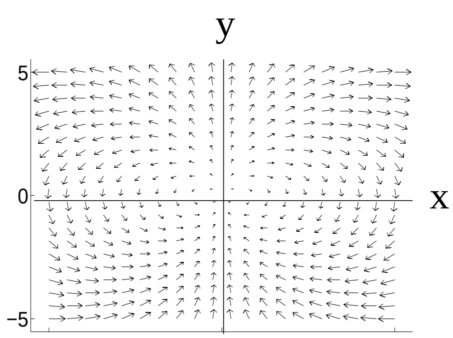
$$\frac{d\mathbf{Y}}{dt} = \mathbf{AY} \quad \text{where} \quad \mathbf{A} = \begin{pmatrix} 1 & -2 \\ 2 & -3 \end{pmatrix}$$

9. (9 points) Listed below are three systems of differential equations and four vector fields. Please match each system with its corresponding vector field (one vector field will not have a corresponding equation). Make sure you justify your answer.

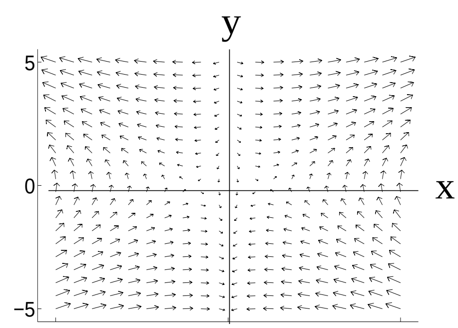
$$\text{a)} \quad \frac{dx}{dt} = \sin(y), \qquad \frac{dy}{dt} = x - x^3$$

$$\text{b)} \quad \frac{dx}{dt} = 2xy, \qquad \frac{dy}{dt} = y^2 - x^2$$

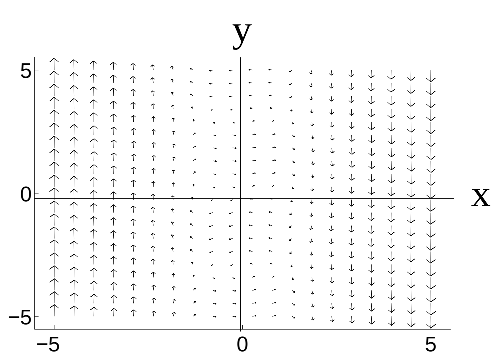
$$\text{c)} \quad \frac{dx}{dt} = 2xy, \qquad \frac{dy}{dt} = x^2 - 1$$



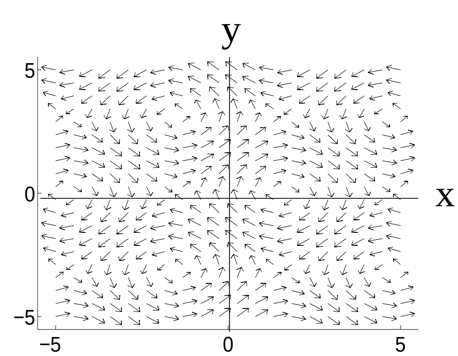
i)



ii)



iii)



iv)

10. Extra Credit - The following two questions are extra credit. Each is worth 5 points, and no partial credit will be given. (It is possible to get credit for one, and not the other.)

a) Consider the linear system $\frac{d\mathbf{Y}}{dt} = \mathbf{AY}$. Suppose the matrix has eigenvalue $\lambda_1 = -3$ with associated eigenvector $\mathbf{V}_1 = (1, 2)$ and eigenvalue $\lambda_2 = -1$ with associated eigenvector $\mathbf{V}_2 = (1, -3)$. Prove that most solution curves approach the origin tangent to the line $y = -3x$.

b) For the following system of differential equations, classify the equilibrium point at the origin for all values of the parameter b.

$$\frac{dx}{dt} = -y + bx^3$$

$$\frac{dy}{dt} = x + by^3$$

Hint: Be careful! Try to find a Lyapunov function for the system.