Nonlinear convective stability of travelling fronts near Turing and Hopf instabilities

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Motivation: Ways to have a Hopf bifurcation

ODEs: Hopf bifurcation via eigenvalues

- Before bifurcation: \( u(x) = 0 \) stable
- After bifurcation: \( u(x) = 0 \) unstable
- After bifurcation: nearby stable periodic orbit exists
- After bifurcation: solutions approach periodic orbit, amplitude saturates
Motivation: Ways to have a Hopf bifurcation

PDEs: Hopf bifurcation via eigenvalues or essential spectrum

- Before bifurcation: equilibrium is stable
- After bifurcation: equilibrium is linearly unstable
- After bifurcation: do nearby stable solutions exist?
- After bifurcation: what state do solutions approach?
"Essential" Hopf bifurcations

What causes an "essential" Hopf bifurcation?

Roughly speaking:

- Eigenvalues: localized perturbations and interior of wave
- Essential spectrum: non-localized perturbations and end states of wave

An "essential" Hopf bifurcation is caused by a destabilization of the end states.
Essential instabilities of fronts: results of Sandstede and Scheel ’01

What patterns can form through essential Hopf instabilities of fronts?

\[ u_*(\xi; \mu) \]

\[ \xi = x - c_* t \]

Two cases:

- Rest state ahead of front destabilizes: \( u_+ \)
- Rest state behind front destabilizes: 0

Expect fronts connecting remaining stable state to emergent patterns:
Essential instabilities of fronts: results of Sandstede and Scheel '01

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Essential instabilities of fronts: ecological examples

Patterns in wake of front in predator-prey models (matches field studies):

[From papers by Jonathan A. Sherratt and colleagues]

In all cases the front outruns the pattern.
Essential instabilities of fronts: results of Sandstede and Scheel ’01

- Destabilization ahead: patterns exist and are stable!
- Destabilization behind: patterns do not exist; front outruns it!

Sandstede and Scheel explain this using exponential dichotomies and Fredholm theory: roughly speaking, dimension counting of stable and unstable manifolds.

Question: when no emergent pattern exists, what is the “stable” behavior?

- Front becomes linearly unstable
- Front is still be observed: must be nonlinearly stable
Set up and assumptions

Reaction-diffusion system

\[ u_t = D \partial_x^2 u + f(u; \mu), \quad u \in \mathbb{R}^n, \quad x \in \mathbb{R}, \quad \mu \approx 0 \]  
\text{(RD)}

**Hypothesis 1: Existence of front solution** \( \forall \mu \approx 0 \)

\[ u(x, t) = u_*(x - c(\mu)t; \mu), \quad c(\mu) > 0 \]

\[ \lim_{\xi \to -\infty} u_*(\xi; \mu) = 0, \quad \lim_{\xi \to +\infty} u_*(\xi; \mu) = u_+ \]

Linearized operator at critical front:

\[ \mathcal{L}_* = D \partial_\xi + c_* \partial_\xi + f_u(u_0^*(\xi); 0), \quad \xi = x - c_* t, \quad c_* = c(0) \]

Asymptotic operators:

\[ \mathcal{L}_-(\mu) = D \partial_x^2 + f_u(0; \mu), \quad \mathcal{L}_+(0) = D \partial_x^2 + f_u(u_+; 0), \]

Exponential weight:

\[ \rho_a(\xi) = \begin{cases} 
1 & \text{if } \xi \geq 1 \\
e^{a\xi} & \text{if } \xi \leq -1
\end{cases} \]
Set up and assumptions

**Hypothesis 2: Spectral assumptions**

- $0 < a \leq a_0$: the spectrum of $L_*^a := \rho_a L_* \rho_a^{-1}$ is in open left half plane except isolated eigenvalue at 0.
- For $\mu \approx 0$, spectrum of $L_-(\mu)$ is in open left half plane except for
  \[
  \lambda(k, \mu) = \lambda_0(\mu) - \lambda_2(\mu)(k - k_0)^2 + \mathcal{O}(|k - k_0|^3), \quad |k - k_0| \ll 1
  \]
  and its complex conjugate, where $\Re \lambda_2(0) > 0$, $\Re \lambda'_0(0) > 0$ and either
  - Turing: $k_0 > 0$ and $\lambda_0(0) = 0$, or
  - Hopf: $k_0 = 0$, $\lambda_0(0) = i\omega_0$ for some $\omega_0 > 0$.
- The spectrum of $L_+(0)$ lies in the open left half plane.

Note: Picture is in moving frame $\xi$, so $\omega_0 = k_0 c_* > 0$ at Turing bifurcations.
Set up and assumptions

Need bifurcation to be supercritical:

\[ u(x, t) = \epsilon e^{i(k_0 x + \omega_0 t)} A(\epsilon x, \epsilon^2 t) e(k_0) + \text{c.c.}, \quad \mu = \rho \epsilon^2 \]

Amplitude \( A(X, T) \) satisfies Ginzburg-Landau equation

\[ A_t = \lambda_2(0) \partial_X^2 A + \rho \lambda'_0(0) A - b |A|^2 A \]

**Hypothesis 3: Supercritical bifurcation**

\[ \Re b > 0 \]

This ensures the growth of the emergent pattern saturates.

Function space: uniformly local functions

\[ \rho_{ul}(x) = e^{-|x|}, \quad \|u\|_{\rho_{ul}}^2 = \int_{\mathbb{R}} \rho_{ul}(x) |u(x)|^2 \, dx \quad \|u\|_{L_{ul}^2} = \sup_{y \in \mathbb{R}} \|u\|_T y \rho_{ul}. \]

Like normal Sobolev spaces but allow for nonlocalized functions.
Statement of result

**Theorem** [B., Ghazaryan, Sandstede JDE 09] Assume (H1)-(H3), then there exist positive constants $K$, $\Lambda_*$, $a_*$, $\mu_*$, and $\delta_*$ such that: for any

$$|\mu| \leq \mu_*, \quad \|v(\cdot, 0)\|_{H_{u_1}^1} < \delta_*, \quad \|v(\cdot, 0)\|_{H_{u_1}^1} < \delta_*,$$

the solution of (RD) with $u(x, 0) = u_*(x; \mu) + v(x, 0)$ exists for all $t \geq 0$ and

$$u(x, t) = u_*(x - c(\mu)t - p(t); \mu) + v(x - c(\mu)t, t)$$

for an appropriate real-valued function $p$; furthermore, $\exists p_* \in \mathbb{R}$ such that

$$\|v(\cdot, t)\|_{H_{u_1}^1} + |p(t)| \leq K \left( \|v(\cdot, 0)\|_{H_{u_1}^1} + \sqrt{|\mu|} \right)$$

$$\|\rho_{a_*} (\cdot) v(\cdot, t)\|_{H_{u_1}^1} + |p(t) - p_*| \leq Ke^{-\Lambda_* t}$$

for all $t \geq 0$.

In other words, the perturbation $v(\xi, t)$ decays to zero exponentially in time in the weighted norm $\|\rho_{a_*} \cdot\|_{H_{u_1}^1}$ in the comoving frame $\xi = x - c(\mu)t$. 
Intuition

Front outruns emergent pattern

and because bifurcation is supercritical, growth of pattern saturates.

[From Sandstede and Scheel, Dynamical Systems Vol. 16, 2001]
Difficulties in proof

Just after bifurcation:

Mathematical issues:

• Need to control the growth this causes
• No spectral gap: how to isolate this growth?

Resolution: mode filters

• Developed by G. Schneider, 1994 papers
• Generalization of a spectral projection
Mode filters

Standard spectral projection:

\[ e^{\mathcal{L}t} = \frac{1}{2\pi i} \int_{\Gamma_1} e^{\lambda t} (\lambda - \mathcal{L})^{-1} d\lambda + \frac{1}{2\pi i} \int_{\Gamma_2} e^{\lambda t} (\lambda - \mathcal{L})^{-1} d\lambda \]

\[ = e^{\mathcal{L}t} P_c + e^{\mathcal{L}t} P_s \]

Mode filters effectively allow for:

\[ e^{\mathcal{L}t} = e^{\mathcal{L}t} P_{mf}^c + e^{\mathcal{L}t} \left( 1 - P_{mf}^c \right) \]

\[ P_{mf}^s \]
Proof: two steps

\[ u(x, t) = u^* (x - c(\mu)t - p(t); \mu) + v(x - c(\mu)t, t) \]

**Step 1:** A prior estimates in weighted space:

If \[ \|v(t)\|_{H^1_{ul}} < \delta \quad \forall t \geq 0 \]

Then \[ \|\rho_a v(t)\|_{H^1_{ul}} \leq Ke^{-\Lambda^* t} \quad \forall t \geq 0 \]

**Step 2:** Use mode filters to prove that, indeed

\[ \|v(t)\|_{H^1_{ul}} < \delta \quad \forall t \geq 0 \]

**Note:** strategy similar to [Ghazaryan & Sandstede '07], but they had a specific system and their step 2 was proved using the maximum principle.
Step 1: A priori estimates in weighted space

Expect: \( u(\xi, t) \rightarrow u_*(\xi - p) \) as \( t \to \infty \).

\[
\begin{align*}
  u(x, t) &= u_*(x - c(\mu)t - p(t); \mu) + v(x - c(\mu)t, t) \\
  &= u_*(\xi - p(t)) + v(\xi, t)
\end{align*}
\]

Unshifted linearized operator:

\[
\mathcal{L}_0 = D\partial_\xi^2 + c(\mu)\partial_\xi + f_u(u_*(\xi))
\]

Work in weighted space: \( w(\xi, t) := \rho_a(\xi)v(\xi, t) \)

\[
w_t = \mathcal{L}_0^a w + F(p, \xi)w + N(v, w)
\]

Because in weighted space there is a spectral gap:

\[
\begin{align*}
  w_t &= P^s [\mathcal{L}_0^a w + F(p, \xi)w + N(v, w)] \\
  p_t &= CP^c \left[ \tilde{F}(p, \xi)w + N(v, w) \right]
\end{align*}
\]
Step 1: A priori estimates in weighted space

Given $\eta$ sufficiently small, define $T_{\text{max}}(\eta) > 0$ to be the maximum time so that

$$\sup_{t \in [0, T]} \left( |p(t)| + \|v(\cdot, t)\|_{H^1_{ul}} \right) \leq \eta$$

**Lemma** There exists a $\Lambda$ so that if $w$ is a solution in the weighted space, then

$$\|w(\cdot, t)\|_{H^1_{ul}} \leq Ke^{-\Lambda t} \|w(\cdot, 0)\|_{H^1_{ul}}, \quad |p(t)| \leq K \|w(\cdot, 0)\|_{H^1_{ul}}$$

for all $0 \leq t \leq T_{\text{max}}(\eta)$, for some positive constant $K$ that is independent of $\mu$ and $\eta$. If $T_{\text{max}}(\eta) = \infty$, then there is a $p_\ast \in \mathbb{R}$ with

$$|p(t) - p_\ast| \leq Ke^{-\Lambda t} \|w(\cdot, 0)\|_{H^1_{ul}}$$

for all $t \geq 0$.

Therefore, in step 2 we need to show $T_{\text{max}}(\eta) = \infty$. 
Step 2: estimates in unweighted space via mode filters

**Proposition** There exist positive constants $K$, $\delta_*$ and $\mu_*$ such that, if $\|v(0)\|_{H^1_{ul}} < \delta_*$, then for each $\mu$ with $|\mu| \leq \mu_*$ the perturbation satisfies

$$\|v(\cdot, t)\|_{H^1_{ul}} + |p(t)| \leq K \left( \|v(\cdot, 0)\|_{H^1_{ul}} + \sqrt{|\mu|} \right)$$

for all $t \geq 0$. In particular, $T_{\text{max}}(\eta) = \infty$.

Remark: $O(\sqrt{\mu})$ is emergent pattern size due to Ginzburg-Landau theory.

**Proof** This also has two steps:

- Show behavior of perturbations is governed by that at $-\infty$.
- Control behavior at $-\infty$ using mode filters.
Step 2a: behavior at $-\infty$

Strategy: linearize at rest state $0$ and show nothing else matters.

Recall:

$$\mathcal{L}_- = D \partial^2_\xi + c \partial_\xi + f_u(0)$$

Write

$$v_t = \mathcal{L}_- v + \mathcal{N}_-(v) + \Delta(p, v)$$

$\Delta(p, v) =$ difference in (non)linearization about $u_*$ and about $0$

Lemma  For all $0 \leq t \leq T_{\text{max}}(\eta)$,

$$\|\Delta(p, v)(t)\|_{H^1_{ul}} \leq C(\eta, \Lambda) \|w(t)\|_{H^1_{ul}} \leq C(\eta, \Lambda) e^{-\Lambda t} \|w(0)\|_{H^1_{ul}}.$$
Step 2b: control emergent pattern via mode filters

\[ v_t = D \partial_x^2 v + f(v; \mu) + \Delta(p, v) + O(e^{-\Lambda t}) \]

Ginzburg-Landau formalism for

\[ v_t = D \partial_x^2 v + f(v; \mu) \] (RD)

Consider modulated waves of the form

\[ \nu(x, t) = \delta e^{ik_0 x + i\omega_0 t} A(\delta x, \delta^2 t) e(k_0) + c.c. \] (MW)

Dynamics of amplitude \( A(\delta x, \delta^2 t) = A(X, T) \)

\[ A_T = \lambda_2(0) \partial_X^2 A + \frac{\mu \lambda_0'(0)}{\delta^2} A - b|A|^2 A. \] (GL)

To show (GL) really controls the behavior of \( \nu \), we need:

- Approximation: Given solution of (GL) and (MW), \( \exists \) nearby sol’n of (RD)
- Attractivity: Given sol’n of (RD), \( \exists \) a nearby (MW) via (GL)

Note: Both have been shown by Schneider and Mielke for (RD), ie \( \Delta = 0 \)
Step 2b: define mode filters

Linear operator at $-\infty$, where pattern will form:

$$\mathcal{L}_-(\partial_x) = D\partial_x^2 + f_u(0), \quad \hat{\mathcal{L}}_-(ik) = -k^2 D + f_u(0)$$

Turing bifurcation: for any $\mu \approx 0$ and $k \approx k_0 \neq 0$, near critical mode

$$\lambda(k)\hat{\mathcal{L}}_-(ik) = \lambda(k)\hat{e}(k), \quad \hat{\mathcal{L}}^*_-(ik), \quad \langle \hat{e}(k), \hat{e}^*(k) \rangle = 1$$

Mode filter: defined in Fourier space

$$(\hat{P}_{mf}^{\pm}\hat{u})(k) = \hat{\chi}(2(k \mp k_0))\langle \hat{e}^*(k, \mu), \hat{u}(k) \rangle \hat{e}(k, \mu), \quad \hat{P}_{mf}^c = \hat{P}_{mf}^+ + \hat{P}_{mf}^-$$

Note: doesn’t yield projections!

$$P_{mf}^c \circ P_{mf}^c \neq P_{mf}^c$$
Step 2b: new Ansatz via mode filters

Standard Ansatz: (Turing: $k_0 \neq 0$, $\omega_0 = 0$)

$$v(x, t) = \delta e^{ik_0x} A(\delta x, \delta^2 t) e(k_0) + c.c.$$ 

Tells us $v = v(A)$. For attractivity, need better “guess” and other direction!

$$v(x; A) = \delta e^{ik_0x} \mathcal{F}^{-1} [\hat{\chi}(k) \hat{e}(k + k_0) \mathcal{F}(A(\delta x))] + c.c.$$ 

$$A(X; v) = \frac{1}{\delta} e^{-ik_0X/\delta} (p_{mf}^+ v)(X/\delta)$$

Extract critical modes from $v$ to get $A$.

This allows us to:

- Prove both attractivity and approximation, for (mod RD)
- Therefore, $v$ behaves as predicted by $A$
- Amplitude of pattern must saturate

$$\|v(\cdot, t)\|_{H^1_{ul}} \leq K \left( \|v(\cdot, 0)\|_{H^1_{ul}} + \sqrt{|\mu|} \right)$$
Essential Hopf bifurcation caused by rest state behind front:

After bifurcation, intuitively:

- Front becomes linearly unstable
- No other nearby solutions exist [Sandstede & Scheel 01]
- Numerically: front outruns perturbation
- If growth of perturbation saturates, front should be nonlinearly stable
Summary

Proof had two steps:

1) Show decay in exponentially weighted space if pattern growth saturates

2) Show, via mode filters, that growth does indeed saturate

\[ v_t = D\partial_x^2 v + f(v; \mu) + O(e^{-\Lambda t}) \quad \text{(RD)} \]

\[ A_T = \lambda_2(0)\partial_X^2 A + \frac{\mu\lambda_0'(0)}{\delta^2} A - b|A|^2 A \quad \text{(GL)} \]