Enhanced dissipation in fluids models

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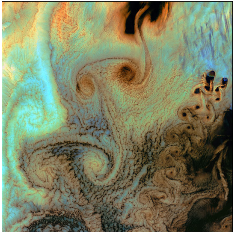
Enhanced dissipation in fluids



Observed in (at least) two models:

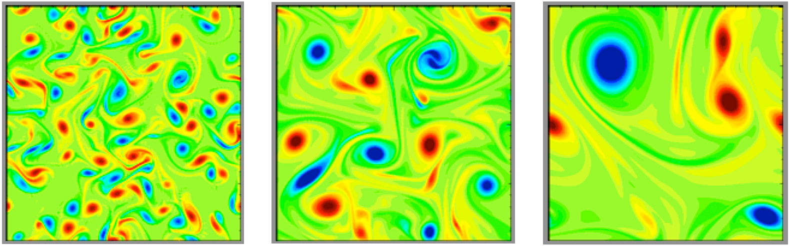
- 2D Navier-Stokes equation
- Taylor dispersion in shear flows

Goal of talk:

- Describe the phenomenology
- Analyze mathematically in above models.

Observed dynamics: 2D Navier-Stokes

2D incompressible Navier-Stokes on the torus with small viscosity:



[Fluid dynamics laboratory, Eindhoven]

- Vorticity evolves from small scale to large scale structures
- Localized vortices persist and organize the dynamics
- Separation of time scales
  - Rapid convergence to localized vortices
  - Slow motion and merger of vortices

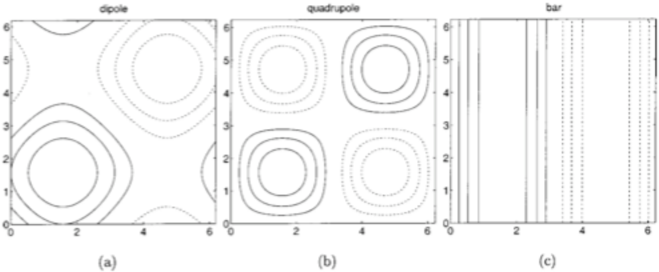
2D Navier-Stokes: decaying turbulence

Some questions:

- How to characterize the quasi-stationary states? [Y, M, C '03]
- What causes the separation in time scales? [B., Wayne '13]

Determine quasi-stationary states via statistical mechanics:

- Stationary solutions of inviscid Euler equations seem to play a role
- Such states with maximum entropy are good candidates

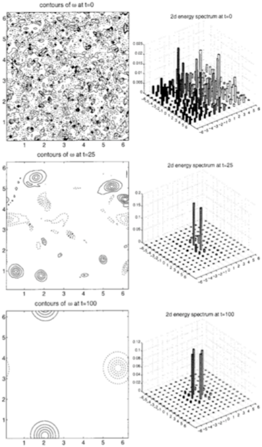


[Yin, Montgomery, Clercx 2003]

Quasi-stationary states

Yin, Montgomery, Clercx 2003:

- Euler: formal calculations and numerical analysis determined these states
- Navier-Stokes: dynamic calculations confirmed predictions ($\nu = 1/5000$)

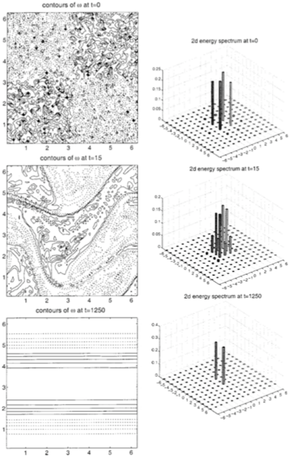


$t = 0$

$t = 25$

$t = 100$

Dipole



$t = 0$

$t = 15$

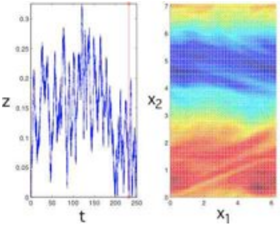
$t = 1250$

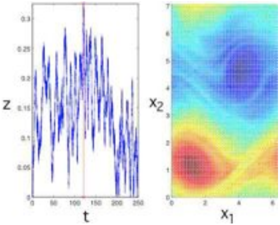
Bar

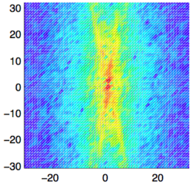
Observed dynamics: stochastically forced Navier-Stokes equation

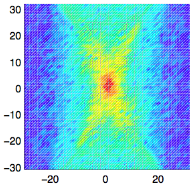
Statistical equilibrium consists of bars and dipoles [Bouchet, Simonnet '09]:

- Square torus: dipole dominates
- Asymmetric (rectangular) torus: bar dominates



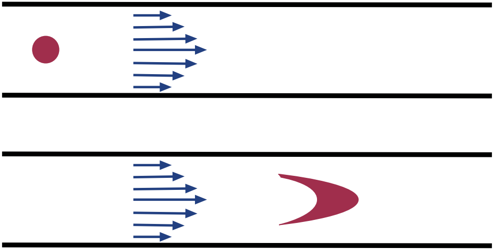






Figures produced by Gabriel Lord (Heriot-Watt)

Observed dynamics: Taylor dispersion in a shear flow



- Named after Geoffrey Taylor (1953)
- Drop dye in pipe with background (nonuniform) shear flow
- Dye will be advected at mean background rate, and also spread out due to both the shear profile and diffusion
- If diffusion coefficient is $0 < \nu \ll 1$, then effective diffusion will be $\sim 1/\nu$!

Analysis: 2D Navier-Stokes on the torus

$$\partial_t \mathbf{u} = \nu \Delta \mathbf{u} - (\mathbf{u} \cdot \nabla)\mathbf{u} - \nabla p, \qquad \nabla \cdot \mathbf{u} = 0, \qquad (x, y) \in \mathbb{T}^2$$

Assume viscosity is small

$$0 < \nu \ll 1, \qquad \text{physical range} = \mathcal{O}(10^{-3}).$$

Vorticity formulation: $\omega = (0, 0, 1) \cdot (\nabla \times \mathbf{u})$

$$\partial_t \omega = \nu \Delta \omega - \mathbf{u} \cdot \nabla \omega, \qquad \int_{\mathbb{T}^2} \omega = 0, \qquad \mathbf{u} = \begin{pmatrix} -\partial_y \Delta^{-1} \omega \\ \partial_x \Delta^{-1} \omega \end{pmatrix}.$$

Decay of energy due to diffusion

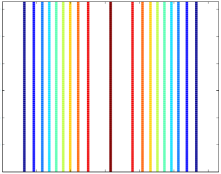
$$\frac{d}{dt} \frac{1}{2} \int_{\mathbb{T}^2} \omega^2(x, y) dxdy = -\nu \int_{\mathbb{T}^2} |\nabla \omega(x, y)|^2 dxdy \leq -\nu \int_{\mathbb{T}^2} \omega^2(x, y) dxdy$$

is very slow

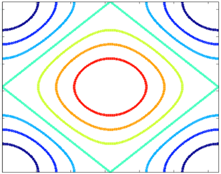
$$\|\omega(t)\|_{L^2} = \mathcal{O}(e^{-\nu t}).$$

Explicit families of metastable states

$$\omega^{bar}(x,y,t) = e^{-\nu t}\cos(x), \qquad \omega^{dipole}(x,y,t) = e^{-\nu t}[\cos(x)+\cos(y)]$$



Bar



Dipole

These solutions:

- Are quasi-stationary if $0 < \nu \ll 1$.
- Match observations of [Yin et al 03] and [Bouchet and Simonnet 09].
- Are stationary solutions of the Euler equations when $\nu = 0$.
- Should attract (some) nearby solutions faster that $\mathcal{O}(e^{-\nu t})$.
- Are part of an infinite family:

$$\omega^{slow}(x,y,t) = e^{-\nu m^2 t}[a_1\cos(mx) + a_2\cos(my) + a_3\sin(mx) + a_4\sin(my)]$$

Linearization about a bar state

$$\partial_t \omega = \nu \Delta \omega - \mathbf{u} \cdot \nabla \omega, \qquad \mathbf{u} = \begin{pmatrix} -\partial_y \Delta^{-1} \omega \\ \partial_x \Delta^{-1} \omega \end{pmatrix}.$$

Ansatz: $\omega = \omega^{bar} + v$

$$\partial_t v = \nu \Delta v - e^{-\nu t}[\sin x \partial_y (1 + \Delta^{-1})] v - \mathbf{u}^v \cdot \nabla v.$$

First understand linear evolution:

$$\partial_t v = \nu \Delta v - e^{-\nu t}[\sin x \partial_y (1 + \Delta^{-1})] v =: \mathcal{L}(t) v$$

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Role of pieces of operator:

- $\nu \Delta$: causes diffusive decay $\mathcal{O}(e^{-\nu t})$
- $e^{-\nu t} \sin x \partial_y \Delta^{-1}$: nonlocal, expect higher order
- Approximation similar to passive scalar advection by shear flow:

$$\partial_t v = \nu \Delta v - \sin x \partial_y v$$

Asymptotic of eigenvalues in [Vanneste, Byatt-Smith 07]: $\mathcal{O}(e^{-\sqrt{\nu} t})$

Expect $\|v(t)\| = \mathcal{O}(e^{-\sqrt{\nu} t}) \ll \mathcal{O}(e^{-\nu t})$. Second term must increase decay!

What causes the fast decay?

$$u_t = Lu$$

Villani, 2009, considers operators of the form

$$L = A^*A + B, \qquad B^* = -B$$

- $\underline{AB = BA}$: antisymmetry of B implies $\|e^{Bt}u\| = \|u\|$, and so

$$\|e^{Lt}\| = \|e^{A^*At}e^{Bt}\| = \|e^{A^*At}\|,$$

so B cannot increase the decay rate of the semigroup.

- $\underline{AB \neq BA}$: rapid decay possible via hypoceorcivity

Define commutator $C = [A, B] = AB - BA$ and a functional

$$\Phi(u) = (u, u) + \alpha(Au, Au) - 2\beta \mathrm{Re}(Au, Cu) + \gamma(Cu, Cu)$$

Careful choice of α, β, and γ can show faster than expected decay.

Back to our problem...

$$\partial_t v = \nu \Delta v - e^{-\nu t}[\sin x \partial_y (1 + \Delta^{-1})] v =: \mathcal{L}(t) v$$

Slow modes: Cannot expect rapid decay on all of L^2

$$\lambda v_{slow} = \partial_t v_{slow} = \mathcal{L}(t) v_{slow}, \qquad \lambda = \mathcal{O}(\nu)$$
$$v_{slow} \in \{e^{-\nu m^2 t + imx}, e^{-\nu t \pm iy} : m \in \mathbb{Z}_0\}.$$

Like an infinite-dimensional eigenspace – need to "project" off it.

Intuitively:

- Expect something like a center manifold with slow decay $\mathcal{O}(e^{-\nu t})$
- and something like a stable manifold with rapid decay $\mathcal{O}(e^{-\sqrt{\nu} t})$
- Use hypocoercivity to get rapid decay rate in stable manifold.
- But operator is time-dependent.
- Can't use spectral projections to obtain manifolds.

Invariant subspaces:

- Can construct them directly by careful inspection.
- Related to movement of energy between Fourier modes.

Rapid decay in "stable" subspace

$$\partial_t v = \nu \Delta v - e^{-\nu t}[\sin x \partial_y (1 + \Delta^{-1})] v.$$

Since there is no y-dependence in the bar state: $v(x, y) = \sum_{l \in \mathbb{Z}} \hat{v}_l(x) e^{ily}$

$$\partial_t \hat{v}_l = \nu \Delta_l \hat{v}_l - ile^{-\nu t}[\sin x (1 + \Delta_l^{-1})] \hat{v}_l, \qquad \Delta_l = \partial_x^2 - l^2.$$

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Recall: want $L = A^* A + B$, with $B^* = -B$

- $A = \partial_x$, $A^* = -\partial_x$, so that $\nu \partial_x^2 = -\nu A^* A$
- But the second term is not anti-symmetric! Change variables...

Motivated by Wilkinson's book "The algebraic eigenvalue problem":

$$u := \sqrt{1 + \Delta_l^{-1}} \hat{v}_l$$

$$\widehat{1 + \Delta_l^{-1}} = 1 - \frac{1}{k^2 + l^2} \qquad \Leftrightarrow \qquad |l| + |k| > 1.$$

Invertible transformation in our subspace.

Transformed equation

$$\partial_t u = \nu \Delta_l u - ile^{-\nu t} \left[\sqrt{1 + \Delta_l^{-1}} \sin x \sqrt{1 + \Delta_l^{-1}} \right] u.$$

We have

- $A := \partial_x$
- $B := -ile^{-\nu t} \left[\sqrt{1 + \Delta_l^{-1}} \sin x \sqrt{1 + \Delta_l^{-1}} \right], \ B^* = -B$
- $C := [\partial_x, B] = -ile^{-\nu t} \left[\sqrt{1 + \Delta_l^{-1}} \cos x \sqrt{1 + \Delta_l^{-1}} \right], \ C^* = -C.$

Problem: $[B, C] \neq 0$; will lead to difficult terms in Villani's framework.

Transformed equation

$$\partial_t u = \nu \Delta_l u - ile^{-\nu t} \left[\sqrt{1 + \Delta_l^{-1}} \sin x \sqrt{1 + \Delta_l^{-1}} \right] u.$$

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Problem: $[B, C] \neq 0$; will lead to difficult terms in Villani's framework.
Partial solution: first consider only the approximate equation

$$\partial_t u = \nu \Delta_l u - ile^{-\nu t} \sin xu := \mathcal{L}_{approx}(t)u.$$

- $A := \partial_x$
- $B := -ile^{-\nu t} \sin x, \ B^* = -B$
- $C := [\partial_x, B] = -ile^{-\nu t} \cos x, \ C^* = -C.$
- $[B, C] = 0$

Analysis: 2D Navier-Stokes, Main result

Function space: $C = C(l) = -ile^{-\nu t}\cos x$

$$X = \left\{ u : \hat{u}_0 = 0, \sum_{l\neq 0}[\|\hat{u}_l\|^2 + \sqrt{\frac{\nu}{|l|}}\|\partial_x \hat{u}_l\|^2 + \frac{1}{\sqrt{\nu}|l^{3/2}}\|C(l)\hat{u}_l\|^2] < \infty \right\}$$

Theorem [B., Wayne '13] Pick $T \in [0, 1/\nu]$. There exist constants K and M, $\mathcal{O}(1)$ with respect to ν, such that the following holds. If ν is sufficiently small, then the solution to $u_t = \mathcal{L}_{approx}(t)u$ with initial condition $u^0 \in X$ satisfies

$$\|u(t)\|_X^2 \leq Ke^{-M\sqrt{\nu}t}\|u^0\|_X^2$$

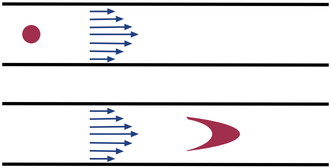
for all $t \in [0, T]$.

Implies rapid decay of solutions:

- Decay $e^{-M\sqrt{\nu}t}$ much faster than the viscous time scale $e^{-\nu t}$
- If $T = 1/\nu$, then

$$e^{-M\sqrt{\nu}T} = e^{-\frac{M}{\sqrt{\nu}}} \ll 1, \qquad e^{-\nu T} = e^{-1}$$

Taylor Dispersion



$$u_t = \nu \Delta u - V(y,z) u_x$$

$$x \in \mathbb{R}, \qquad (y,z) \in \Omega \subset \mathbb{R}^2, \qquad \frac{\partial u}{\partial n}|_\Omega = 0, \qquad 0 < \nu \ll 1$$

Remove background advection

$$V(y,z) = A(1 + \chi(y,z)), \qquad A = \frac{1}{\mathrm{vol}(\Omega)} \int_\Omega V(y,z) \mathrm{d}y\mathrm{d}z.$$

Moving coordinate ($x \to x + At$) and rescale

$$X = \nu x, \qquad T = \nu t.$$

$$u_T = \nu^2 u_{XX} + \Delta_{y,z} u - A\chi(y,z) u_X.$$

Fourier series wrt eigenfunctions $\{\psi_n\}$ of $\Delta_{y,z}$ with eigenvalues $\{\mu_n\}$:

$$u(X,y,z,T) = \sum_{n=0}^{\infty} u_n(X,t)\psi_n(y,z), \qquad \chi(y,z) = \sum_{n=0}^{\infty} \chi_n \psi_n(y,z),$$

to obtain

$$\begin{aligned}
\partial_T u_0 &= \nu^2 \partial_X^2 u_0 - A \sum_{m=1}^{\infty} \chi_m \partial_X u_m \\
\partial_T u_n &= \nu^2 \partial_X^2 u_n - \mu_n u_n - A\chi_n \partial_X u_0 - A \sum_{m=1}^{\infty} \chi_{n,m} \partial_X u_m, \qquad n \neq 0
\end{aligned}$$

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\end{aligned}$$

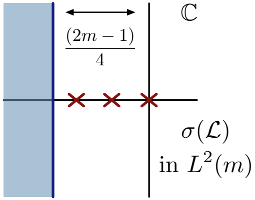
Similarity variables: $\xi = X/\sqrt{1+T}$, $\tau = \log(1+T)$

$$\begin{aligned}
u_0(X,T) &= \frac{1}{\sqrt{1+T}} w_0(\xi,\tau) \\
u_n(X,T) &= \frac{1}{(1+T)} w_n(\xi,\tau), \qquad n \neq 0.
\end{aligned}$$

Taylor Dispersion

Laplacian $\nu^2 \partial_X^2$ becomes in similarity variables:

$$\mathcal{L}w = \nu^2 \partial_\xi^2 w + \frac{1}{2}\partial_\xi(\xi w)$$



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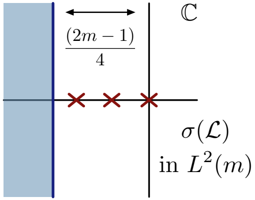
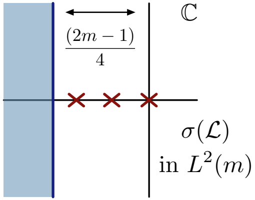


Dynamics of Fourier modes becomes

$$\begin{aligned}
\partial_\tau w_0 &= \mathcal{L}w_0 - A\sum_{m\neq 0}\hat{\chi}_m\partial_\xi w_m \\
\partial_\tau w_n &= \left(\mathcal{L}+\frac{1}{2}\right)w_n - e^{\tau/2}A\sum_{m=1}^{\infty}\chi_{n,m}\partial_\xi w_m - e^{\tau}(\mu_n w_n + A\chi_n\partial_\xi w_0).
\end{aligned}$$

Taylor Dispersion



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For large τ,

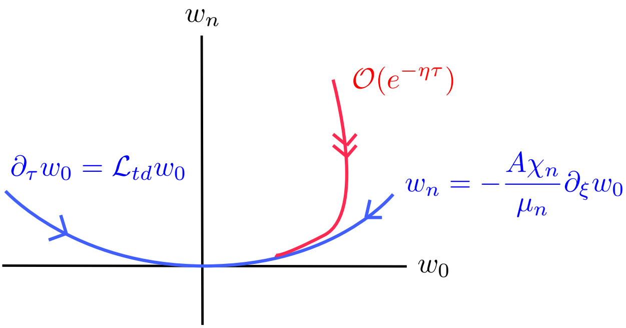
$$w_n \sim -\frac{A\chi_n}{\mu_n}\partial_\xi w_0 \qquad \Rightarrow \qquad \partial_\tau w_0 = \mathcal{L}w_0 + A^2\sum_{m\neq 0}\frac{1}{\mu_m}|\hat{\chi}_m|^2\partial_\xi^2 w_0.$$

Rapid diffusion:

$$\partial_\tau w_0 = \nu_{td}\partial_\xi^2 w_0 + \frac{1}{2}\partial_\xi(\xi w_0), \qquad \nu_{td} = \nu^2 + \underbrace{A^2\sum_{m\neq 0}\frac{1}{\mu_m}|\hat{\chi}_m|^2}_{=:D_{td}}$$

Taylor Dispersion

Intuitively, we expect:



But this is not quite true!

- Enhanced diffusion affects only low modes (isolated eigenvalues)
- Infinite-dimensional ODE governing low modes has a center manifold
- Convergence to the center manifold is exponentially fast
- High modes (rest of spectrum) decay exponentially due to regular diffusion

$$e^{\nu^2 \partial_x^2 t} \sim e^{-\nu^2 k^2 t} \sim e^{-t} \quad \text{if} \quad k \sim 1/\nu$$

- Taylor dispersion only affects low modes, still physically observable.

$$L^2(m) = \left\{ w \in L^2(\mathbb{R}) : \int_{\mathbb{R}} (1+X^2)^m |w(X)|^2 \mathrm{d}X < \infty \right\}$$

Theorem [B., Chaudhary, Wayne '18] Given any N, if $u(\cdot, y, z, 0) \in L^2(N+1)$

$$u(X,y,z,T) = u_{\mathrm{app}}(X,y,z,T) + u_{\mathrm{rem}}(X,y,z,T),$$

where

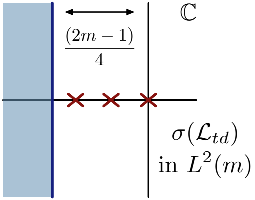
1. u_{app} is governed by an infinite-dimensional system of ODEs that possesses a globally exponentially attracting center manifold on which the dynamics correspond to enhanced diffusion with viscosity ν_{td}. In other words,

$$\left\| u_{\mathrm{app}}(X,y,z,T) - \frac{C}{\sqrt{4\pi\nu_{td}(T+1)}} e^{-\frac{X^2}{4\nu_{td}(T+1)}} \right\|_{L^2} \leq \frac{C}{(1+T)^{3/2}}.$$

2. The remainder term satisfies

$$\|u_{\mathrm{rem}}(X,y,z,T)\|_{L^2} \leq \frac{C}{(1+t)^{\frac{N}{6}+\frac{1}{12}}}.$$

Ideas in Proof



$$\partial_\tau w_0 = \mathcal{L}_{td} w_0 - D_{td}\partial_\xi^2 w_0 - A \sum_{m\neq 0} \hat{\chi}_m \partial_\xi w_m$$

$$\partial_\tau w_n = \left(\mathcal{L}_{td} + \frac{1}{2}\right) w_n - D_{td}\partial_\xi^2 w_0 - e^{\tau/2} A \sum_{m=1}^{\infty} \chi_{n,m}\partial_\xi w_m - e^{\tau}\left(\mu_n w_n + A\chi_n \partial_\xi w_0\right)$$

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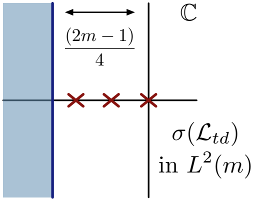

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Separate into low and high modes using $\sigma(\mathcal{L}_{td})$:

$$w_0(\xi,\tau) = \sum_{k=0}^{N}\alpha_k(\tau)\varphi_k^{td}(\xi) + w_0^s(\xi,\tau)$$

$$w_n(\xi,\tau) = \sum_{k=0}^{N}\beta_k^n(\tau)\varphi_k^{td}(\xi) + w_n^s(\xi,\tau), \qquad n = 1,2,3,\ldots$$

Define u_{app} and u_{rem}:

- u_{app} defined via $\alpha_k\varphi_k^{td}$ and $\beta_k^n\varphi_k^{td}$
- u_{rem} defined via w_0^s and w_n^s

Analysis of u_{app}:

$$
\begin{aligned}
w_0(\xi,\tau) &= \sum_{k=0}^{N} \alpha_k(\tau)\varphi_k^{td}(\xi) + w_0^s(\xi,\tau) \\
w_n(\xi,\tau) &= \sum_{k=0}^{N} \beta_k^n(\tau)\varphi_k^{td}(\xi) + w_n^s(\xi,\tau), \qquad n = 1,2,3,\ldots
\end{aligned}
$$

Project equations onto low eigenmodes: $2 \le k \le N$, $n = 1,2,3,\ldots$

$$
\begin{aligned}
\dot{\alpha}_0 &= 0 \\
\dot{\alpha}_1 &= -\frac{1}{2}\alpha_1 \\
\dot{\alpha}_k &= -\frac{k}{2}\alpha_k - D_{td}\alpha_{k-2} - A\sum_{m=1}^{\infty}\chi_m\beta_{k-2}^m \\
\dot{\beta}_0^n &= -e^{\tau}(\mu_n\beta_0^n + A\chi_n\alpha_0) \\
\dot{\beta}_1^n &= -\frac{1}{2}\beta_1^n - e^{\tau}(\mu_n\beta_1^n + A\chi_n\alpha_1) - e^{\frac{\tau}{2}}A\sum_{m=1}^{\infty}\chi_{n,m}\beta_0^m \\
\dot{\beta}_k^n &= -\frac{k}{2}\beta_k^n - e^{\tau}(\mu_n\beta_k^n + A\chi_n\alpha_k) - D_{td}\beta_{k-2}^n - e^{\frac{\tau}{2}}A\sum_{m=1}^{\infty}\chi_{n,m}\beta_{k-1}^m
\end{aligned}
$$

Analysis of u_{app}:

Diagonalize and make autonomous via $\sigma = e^{-\tau/2}$, $\tau = \log(1+T)$:

$$\begin{aligned}
a_0' &= 0\\
a_1' &= -\frac{1}{2}\sigma^2 a_1\\
a_k' &= \sigma^2\left(-\frac{k}{2}a_k - A\sum_{m=1}^{\infty}\chi_m b_{k-2}^m\right)\\
b_0^{n\prime} &= -\mu_n b_0^n\\
b_1^{n\prime} &= -\left(\frac{1}{2}\sigma^2 + \mu_n\right)b_1^n - A\sigma\sum_{m=1}^{\infty}\chi_{n,m}\left(b_0^m - \frac{A\chi_m}{\mu_m}a_0\right)\\
b_k^{n\prime} &= -\left(\frac{k}{2}\sigma^2 + \mu_n\right)b_k^n - D_T\sigma^2\left(b_{k-2}^n - \frac{A\chi_n}{\mu_n}a_{k-2}\right) - \frac{A^2\chi_n}{\mu_n}\sigma^2\sum_{m=1}^{\infty}\chi_m b_{k-2}^m\\
&\quad -\sigma A\sum_{m=1}^{\infty}\chi_{n,m}\left(b_{k-1}^m - \frac{A\chi_m}{\mu_m}a_{k-1}\right)\\
\sigma' &= -\frac{1}{2}\sigma^3,
\end{aligned}$$

Analysis of u_{app}:

Write

$$b_k = \{b_k^n\}_{n=1}^{\infty}$$

Proposition The above system has an invariant center-stable manifold given by

$$\mathcal{M}_N = \{(b_0, \dots, b_N) = (0, h_1(a_0, \sigma), \dots, h_N(a_0, \dots, a_{N-1}, \sigma))\}.$$

Moreover, there exist constants $C, \eta_1, \eta_2 > 0$ so that

$$\|(b_0, \dots, b_N, \gamma)(T) - (0, h_1(a_0, \sigma), \dots, h_N(a_0, \dots, a_{N-1}, \sigma), 0)\|_{(\ell^2)^{N+2}} \le Ce^{-\eta_1 T},$$

while

$$|a_k(T)| \le \frac{C}{(1+T)^{\eta_2}}, \qquad 1 \le k \le N.$$

Remark: The functions h_k can be determined explicitly, as can the rates $\eta_{1,2}$. We essentially have

$$\eta_1 \sim \mu_1, \qquad \eta_2 \sim k/6.$$

Analysis of u_{rem}:

$$
\begin{aligned}
w_0(\xi,\tau) &= \sum_{k=0}^{N} \alpha_k(\tau)\varphi_k^{td}(\xi) + w_0^s(\xi,\tau) \\
w_n(\xi,\tau) &= \sum_{k=0}^{N} \beta_k^n(\tau)\varphi_k^{td}(\xi) + w_n^s(\xi,\tau), \qquad n = 1,2,3,\ldots
\end{aligned}
$$

Project off lowest eigenmodes to obtain equations for w_0^s, w_n^s. Convert back to (X, T) variables and take the Fourier transform in X to obtain

$$
\frac{d}{dT}\hat{U} = \mathcal{B}(\kappa)\hat{U} + \hat{F}(\kappa, T), \qquad \hat{U} = \begin{pmatrix} \hat{u}_0^s(\kappa, T) \\ \{\hat{u}_n^s(\kappa, T)\}_{n=1}^{\infty} \end{pmatrix}
$$

where

$$
\mathcal{B}(\kappa) = -\nu^2\kappa^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \mathrm{i}\kappa A \begin{pmatrix} 0 & \chi\cdot \\ \chi & \chi* \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & \Upsilon \end{pmatrix}
$$

and

$$
\Upsilon = \mathrm{diag}(\mu_n), \qquad n = 1,2,3,\ldots
$$

Analysis of u_{rem}:

$$\frac{d}{dT}\hat{U} = \mathcal{B}(\kappa)\hat{U} + \hat{F}(\kappa, T), \qquad \hat{U} = \begin{pmatrix} \hat{u}_0^s(\kappa, T) \\ \{\hat{u}_n^s(\kappa, T)\}_{n=1}^{\infty} \end{pmatrix}$$

where

$$\mathcal{B}(\kappa) = -\nu^2\kappa^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \mathrm{i}\kappa A \begin{pmatrix} 0 & \chi\cdot \\ \chi & \chi* \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & \Upsilon \end{pmatrix}$$

Prove rapid decay in T by considering three regions:

- $|\kappa| \le \kappa_0$: Spectrum divided into $\lambda_{td}(\kappa)$ and $\Sigma(\kappa)$
  - $\lambda_{td}(\kappa)$ only causes $T^{-N/6}$ decay because of definition of $\hat{U}$
  - $\Sigma(\kappa)$ gives decay like $e^{-\mu_1 T}$.
- $\kappa_0 \le |\kappa| \le C/\nu$: use hypocoercivity estimate to obtain exponential decay.
- $C/\nu \le |\kappa|$: exponential decay via usual diffusive estimate

$$e^{\nu^2\partial_X^2 t} \sim e^{-\nu^2\kappa^2 t} \sim e^{-t} \quad \text{if} \quad \kappa \sim 1/\nu$$

Summary

Fluids can exhibit enhanced decay due to diffusion.

2D Navier-Stokes on the torus with small viscosity:

- Rapid convergence to bar states $\mathcal{O}(e^{-\sqrt{\nu}t})$
- Slow diffusive decay to rest state $\mathcal{O}(e^{-\nu t})$
- Analyzed approximate operator using the theory of hypocoercive operators

Model of shear flow to study Taylor Dispersion:

- Enhanced diffusion affects only low modes
- Intermediate Fourier modes decay exponentially fast via hypocoercivity
- High Fourier modes decay exponentially fast due to usual diffusion
- Evolution of low Fourier modes and Taylor diffusion can be explained using similarity variables and invariant manifolds

Analysis: Towards the stochastic 2D Navier-Stokes equation [B. Cooper, Spiliopoulos]

Vorticity formulation of 2D Navier-Stokes:

$$\partial_t \omega = \nu \Delta \omega - \mathbf{u} \cdot \nabla \omega, \qquad \mathbf{u} = \begin{pmatrix} -\partial_y \Delta^{-1} \omega \\ \partial_x \Delta^{-1} \omega \end{pmatrix}.$$

Above-mentioned results suggest:

- Bar states and Dipoles are important; correspond to low Fourier modes
- Any asymmetry (ie square vs rectangular) in the torus is important
- Transfer of energy between Fourier modes is important

Work on $(x, y) \in \Omega = [0, 2\pi\delta] \times [0, 2\pi]$ with periodic boundary conditions:

$$\omega(x, y) = \sum_{k \neq 0} \hat{\omega}(k, l) e^{\mathrm{i}(kx/\delta + ly)}, \qquad \hat{\omega}(k, l) = \frac{1}{4\pi^2\delta} \int_\Omega \omega(x, y) e^{-\mathrm{i}(kx/\delta + ly)} \mathrm{d}x \mathrm{d}y.$$

- Bar states: $\hat{\omega}(\pm 1, 0)$ or $\hat{\omega}(0, \pm 1)$ nonzero, rest zero
- Dipoles: combination of $\hat{\omega}(\pm 1, 0)$ and $\hat{\omega}(0, \pm 1)$ nonzero, rest zero

Analysis: create model problem via Center Manifold Reduction

Formal Center Manifold Reduction:

- Include lowest four modes to capture bars/dipoles
- Include some higher to model energy transfer between low/high modes
- Simplest reasonable model: lowest 8 modes

$$\begin{array}{ll} \text{low}: & w_1 = \hat{\omega}(1,0) \quad w_2 = \hat{\omega}(-1,0) \quad w_3 = \hat{\omega}(0,1) \quad w_4 = \hat{\omega}(0,-1) \\ \text{“high”}: & w_5 = \hat{\omega}(1,1) \quad w_6 = \hat{\omega}(-1,1) \quad w_7 = \hat{\omega}(1,-1) \quad w_8 = \hat{\omega}(-1,-1) \end{array}$$

We obtain

$$\frac{d}{dt} W = F(W; \nu, \delta) = \mathcal{O}(|W|, |W|^2), \qquad W = (w_1, \dots, w_8).$$

Note: the construction of this center manifold is local, and possibly holds only in a small, $\mathcal{O}(\nu)$, neighborhood of $\omega = 0$!

Results for model ODE

$$\frac{d}{dt}W = F(W;\nu,\delta), \qquad W = (w_1,\dots,w_8).$$

Theorem [B., Cooper, Spiliopoulos '17]

- For all δ sufficienlty close to one, the high modes decay at the rate $\mathcal{O}(e^{-t/\nu})$, while the low modes decay at the rate $\mathcal{O}(e^{-\nu t})$.
- On the square torus, when $\delta = 1$, most initial conditions will evolve to a dipole state.
- On the asymmetric torus, when $\delta \neq 1$, most initial conditions will evolve to a bar state. It will be an x-bar state if $\delta > 1$, and a y-bar state if $\delta < 1$.

Proof Methods:

- Analysis for $\delta = 1$ is much easier because many terms drop out.
- Straightforward energy estimates show background decay of $\mathcal{O}(e^{-\nu t})$
- Detailed estimates on transient timescales show fast decay of high modes.
- Convergence to bar/dipole determined by evolution of

$$R(t) = \frac{|w_1(t)|^2}{|w_3(t)|^2}$$

Future work: add noise to ODE model to study stochastic 2D Navier Stokes

Analysis: create model problem via Center Manifold Reduction

Using the relation $\hat{\omega}(k, l) = \bar{\hat{\omega}}(-k, -l)$, we have

$$
\begin{aligned}
\dot{w}_1 &= -\frac{\nu}{\delta^2} w_1 + \frac{1}{\delta(1+\delta^2)}[w_3 w_7 - \bar{w}_3 w_5] + \frac{3\delta^6}{2\nu(4+\delta^2)(1+\delta^2)^2} w_1 [|w_5|^2 + |w_7|^2] \\
\dot{w}_3 &= -\nu w_3 + \frac{\delta^3}{(1+\delta^2)}[\bar{w}_1 w_5 - w_1 \bar{w}_7] + \frac{3\delta^6}{2\nu(1+4\delta^2)(1+\delta^2)^2} w_1 [|w_5|^2 + |w_7|^2] \\
\dot{w}_5 &= -\nu \frac{(1+\delta^2)}{\delta^2} w_5 - \frac{(\delta^2-1)}{\delta} w_1 w_3 + \frac{9\delta^5(\delta^2-1)}{4\nu^2(4+\delta^2)(1+4\delta^2)(1+\delta^2)^2} w_1 w_3 |w_7|^2 \\
&\quad - \frac{(1+3\delta^2)}{2\nu\delta^2(1+4\delta^2)(1+\delta^2)} w_5 |w_3|^2 - \frac{\delta^6(3+\delta^2)}{2\nu(4+\delta^2)(1+\delta^2)} w_5 |w_1|^2 \\
\dot{w}_7 &= -\nu \frac{(1+\delta^2)}{\delta^2} w_7 + \frac{(\delta^2-1)}{\delta} w_1 \bar{w}_3 - \frac{9\delta^5(\delta^2-1)}{4\nu^2(4+\delta^2)(1+4\delta^2)(1+\delta^2)^2} w_1 \bar{w}_3 |w_5|^2 \\
&\quad - \frac{(1+3\delta^2)}{2\nu\delta^2(1+4\delta^2)(1+\delta^2)} w_7 |w_3|^2 - \frac{\delta^6(3+\delta^2)}{2\nu(4+\delta^2)(1+\delta^2)} w_7 |w_1|^2
\end{aligned}
$$

Equation for $R = |\omega_1|^2 / |\omega_3|^2$:

$$\dot{R} = -2\nu \left(\frac{1-\delta^2}{\delta^2} \right) R + \text{nonlinear stuff}$$

If $\delta < 1$, $R \to 0$ and evolution is to a y-bar state.

Construct invariant subspaces

$$v(x,y) = \sum_{k,l \in \mathbb{Z},(k,l)\neq(0,0)} \hat{v}(k,l) e^{i(kx+ly)}$$

Goal: don't excite the slow modes

$$\{e^{-\nu m^2 t + imx}, e^{-\nu t \pm iy}\} \quad \Rightarrow \quad (k,l) \in \{(0,\pm 1),(m,0)\}$$

In Fourier space, $v_t = \nu \Delta v - e^{-\nu t}\partial_y \sin x(1+\Delta^{-1})v$ becomes

$$\begin{aligned}\partial_t \hat{v}(k,l) &= -\nu(k^2+l^2)\hat{v}(k,l) \\ &\quad -\frac{l}{2}e^{-\nu t}\left[\left(1-\frac{1}{(k-1)^2+l^2}\right)\hat{v}(k-1,l) - \left(1-\frac{1}{(k+1)^2+l^2}\right)\hat{v}(k+1,l)\right]\end{aligned}$$

Try $\mathcal{M}_x = \{v \in L^2(\mathbb{T}^2) : \hat{v}(m,0) = 0,\ m \in \mathbb{Z}\}$

$$\partial_t \hat{v}(m,0) = -\nu m^2 \hat{v}(m,0) \qquad \textbf{invariant}$$

Try: $\tilde{\mathcal{M}}_y = \{v \in L^2(\mathbb{T}^2) : \hat{v}(0,\pm 1) = 0\}$

$$\partial_t \hat{v}(0,\pm 1) = -\nu \hat{v}(0,\pm 1) \mp \frac{1}{4}e^{-\nu t}\left[\hat{v}(-1,\pm 1) - \hat{v}(1,\pm 1)\right] \qquad \textbf{not invariant}$$

Construct invariant subspaces

Recall: we don't want to excite the modes $e^{\pm imx}$ and $e^{\pm iy}$

- x-modes: $\mathcal{M}_x = \{w \in L^2(\mathbb{T}^2) : \hat{w}(m,0) = 0\}$
- y-modes: Formal calculations with Fourier equation lead to...

Define

$$p_j^{\pm} := \hat{w}(2j, \pm 1) + \hat{w}(-2j, \pm 1), \qquad q_j^{\pm} := \hat{w}(2j+1, \pm 1) - \hat{w}(-2j-1, \pm 1)$$

One can show:

$$\begin{pmatrix} p^{\pm} \\ q^{\pm} \end{pmatrix} = A^{\pm}(t) \begin{pmatrix} p^{\pm} \\ q^{\pm} \end{pmatrix}$$

Propositon A solution of $w_t = \mathcal{L}(t)w$ satisfies $\hat{w}(0, \pm 1)(t) = 0$ for all $t \geq 0$ if and only if $w(0) \in \mathcal{M}_y$, where

$$\mathcal{M}_y = \{w \in L^2 : p_j^{\pm} = q_j^{\pm} = 0 \ \forall j\}.$$

Recall: In [YCM '03], only special initial data converge rapidly to bar states.

Why is this new inner product useful?

Motivated by work of Gallagher, Gallay, and Nier 2009, we rescale time:

$$\partial_t u = (\partial_x^2 - l^2)u + \frac{1}{\nu} Bu.$$

Define, for $(u, u) = \|u\|_{L^2}^2$, $\alpha, \beta, \gamma > 0$,

$$\Phi(t) := (u, u) + \alpha(\partial_x u, \partial_x u) - 2\beta \mathrm{Re}(\partial_x u, Cu) + \gamma(Cu, Cu)$$

If $\beta^2 < \alpha\gamma/4$, Young's inequality implies

$$\|u\|^2 + \frac{\alpha}{2}\|u_x\|^2 + \frac{\gamma}{2}\|Cu\|^2 < \Phi < \|u\|^2 + \frac{3\alpha}{2}\|u_x\|^2 + \frac{3\gamma}{2}\|Cu\|^2.$$

Therefore, by controlling the dynamics of Φ, we can control the above norm.

Strategy:

- Compute $d\Phi/dt$
- Chose α, β, γ to obtain a decay estimate
- Show this implies rapid convergence of solutions to the bar states

Proof of Theorem

$$\partial_t u = (\partial_x^2 - l^2)u + \frac{1}{\nu}Bu$$
$$\Phi(t) := (u,u) + \alpha(\partial_x u, \partial_x u) - 2\beta\mathrm{Re}(\partial_x u, Cu) + \gamma(Cu, Cu)$$

Differentiate:

$$\begin{aligned}\frac{d}{dt}\Phi(t) = {}& [(u_t, u) + (u, u_t)] + \alpha[(\partial_x u_t, \partial_x u) + (\partial_x u, \partial_x u_t)] \\ & -2\beta\mathrm{Re}[(\partial_x u_t, Cu) + (\partial_x u, Cu_t)] + \gamma[(Cu_t, Cu) + (Cu, Cu_t)] \\ & +\gamma[(C_t u, Cu) + (Cu, C_t u)].\end{aligned}$$

The first term gives

$$\begin{aligned}(u_t, u) + (u, u_t) &= ((-l^2 + \partial_x^2 + \frac{1}{\nu}B)u, u) + (u, (-l^2 + \partial_x^2 + \frac{1}{\nu}B)u) \\ &= -2l^2\|u\|^2 - 2\|u_x\|^2 + \frac{1}{\nu}\underbrace{[(Bu, u) + (u, Bu)]}_{=0}\end{aligned}$$

by the anti-symmetry of B.

Proof of Theorem

$$\partial_t u = (\partial_x^2 - l^2)u + \frac{1}{\nu}Bu$$
$$\Phi(t) := (u,u) + \alpha(\partial_x u, \partial_x u) - 2\beta\mathrm{Re}(\partial_x u, Cu) + \gamma(Cu, Cu)$$

The α term gives

$$\begin{aligned}(\partial_x u_t, \partial_x u) + (\partial_x u, \partial_x u_t) &= (\partial_x(-l^2 + \partial_x^2 + \frac{1}{\nu}B)u, u_x) \\ &\quad +(u_x, \partial_x(-l^2 + \partial_x^2 + \frac{1}{\nu}B)u) \\ &= -2l^2\|u_x\|^2 - 2\|u_{xx}\|^2 \\ &\quad +\frac{1}{\nu}[(\partial_x(Bu), u_x) + (u_x, \partial_x(Bu))]\end{aligned}$$

We can bound

$$\begin{aligned}[(\partial_x(Bu), u_x) + (u_x, \partial_x(Bu))] &= (Bu_x, u_x) + (\overbrace{[\partial_x, B]}^{=C}\, u, u_x) \\ &\quad +(u_x, Bu_x) + (u_x, [\partial_x, B]u) \\ &= 2\mathrm{Re}(u_x, Cu) \\ &\leq 2\|u_x\|\|Cu\|.\end{aligned}$$

Proof of Theorem

$$\partial_t u = (\partial_x^2 - l^2)u + \frac{1}{\nu} B u$$

$$\Phi(t) := (u, u) + \alpha(\partial_x u, \partial_x u) - 2\beta \mathrm{Re}(\partial_x u, Cu) + \gamma(Cu, Cu)$$

The β term gives

$$\begin{aligned}(\partial_x u_t, Cu) + (\partial_x u, Cu_t) &= -2l^2 \mathrm{Re}(\partial_x u, Cu) + [(u_{xxx}, Cu) + (u_x, Cu_{xx})] \\ &\quad + \frac{1}{\nu}[(\partial_x(Bu), Cu) + (u_x, C(Bu))]\end{aligned}$$

One can show

$$(\partial_x(Bu), Cu) + (u_x, C(Bu)) = \|Cu\|^2 + (u_x, [C, B]u) = \|Cu\|^2$$

Important term: $-(2\beta/\nu)\|Cu\|^2$

The γ and C_t terms are similar.

Proof of Theorem

Collecting these estimates, we have shown

$$\begin{aligned}\frac{d}{dt}\Phi(t) \quad &\leq \quad -2l^2\|u\|^2 - [2+2\alpha l^2]\|u_x\|^2 - 2\alpha\|u_{xx}\|^2 \\ &\quad + \left(\frac{2\alpha}{\nu} + 2\beta(2l^2+1+\nu)\right)\|u_x\|\|Cu\| + 4\beta\|u_{xx}\|\|Cu_x\| \\ &\quad - \left((2l^2+2)\gamma + \frac{2\beta}{\nu} - 2\gamma\nu\right)\|Cu\|^2 - 2\gamma\|Cu_x\|^2 + 2\gamma\|Bu\|^2.\end{aligned}$$

We now use the fact that $2ab \leq a^2 + b^2$ and scale the parameters as

$$\alpha = \sqrt{\nu}\alpha_0, \qquad \beta = \beta_0, \qquad \gamma = \frac{1}{\sqrt{\nu}}\gamma_0$$

With appropriate conditions on $\alpha_0, \beta_0, \gamma_0$, this gives

$$\frac{d}{dt}\Phi(t) \leq -2\|u\|^2 + 2\frac{\gamma_0}{\sqrt{\nu}}\|Bu\|^2 - \frac{1}{4}\|u_x\|^2 - \frac{3\beta_0}{2\nu}\|Cu\|^2$$

Goal: Show $\Phi' \leq -(M/\sqrt{\nu})\Phi$

Proof of Theorem

$$\|u\|^2 + \frac{\alpha_0\sqrt{\nu}}{2}\|u_x\|^2 + \frac{\gamma_0}{2\sqrt{\nu}}\|Cu\|^2 < \Phi < \|u\|^2 + \frac{3\alpha_0\sqrt{\nu}}{2}\|u_x\|^2 + \frac{3\gamma_0}{2\sqrt{\nu}}\|Cu\|^2$$

$$\frac{d}{dt}\Phi(t) \leq -2\|u\|^2 + 2\frac{\gamma_0}{\sqrt{\nu}}\|Bu\|^2 - \frac{1}{4}\|u_x\|^2 - \frac{3\beta_0}{2\nu}\|Cu\|^2$$

Proposition If $|l| > 1$, then there exists a constant M_0 such that, for all $0 < t < T$,

$$\frac{1}{8}\|u_x\|^2 + \frac{\beta_0}{2\nu}\|Cu\|^2 \geq \frac{M_0|l|\sqrt{\beta_0}}{\sqrt{\nu}}\|u\|^2 \ .$$

Proof: Follows like a similar result in [Gallagher, Gallay, & Nier '09]. Essentially due to connection with harmonic oscillator:

$$H = a\partial_{xx} + bx^2 \quad \Rightarrow \quad (Hu, u)_{L^2(\mathbb{R})} \geq \sqrt{ab}(u, u)_{L^2(\mathbb{R})}$$

Need to be careful about the role of $|l|$. Also, $M_0 = \mathcal{O}(e^{-\nu t})$. □

This implies (after choosing $\alpha_0, \beta_0, \gamma_0$)

$$\Phi'(t) \leq -\frac{M}{\sqrt{\nu}}\Phi(t)$$