A brief introduction to stability theory for linear PDEs

Margaret Beck

June 5, 2012

Abstract

These are notes related to a 4-lecture minicourse given during June 10-11, 2012, at a workshop just preceeding the SIAM conference on Nonlinear Waves and Coherent Structures in Seattle, WA, USA. The title of the workshop was "The stability of coherent structures and patterns," and these four lectures concern stability theory for linear PDEs. The two other parts of the workshop are "Using AUTO for stability problems," given by Björn Sandstede and David Lloyd, and "Nonlinear and orbital stability," given by Walter Strauss.

We will focus on one particular method for obtaining linear stability: proving decay of the associated semigroup. Our strategy will be to state the most relevant theorems from semigroup and spectral theory, indicate where in the literature the proofs can be found, and look at some related examples. We will then see how to apply these theorems to reaction-diffusion equations. In particular, we will show that for reaction-diffusion equations, linear stability can be determined simply by computing the spectrum of the associated linearized operator.

1 Introduction

The purpose of this workshop is to understand some issues related to the stability theory for solutions to PDE. Stability of a particular solution of interest (eg a travelling wave or another type of coherent structure or pattern) means, roughly speaking, that if the system starts with an initial condition near that particular solution, then the system will stay near it for all time.

Generally speaking we will focus on systems that can be written in the form

$$u_t = Lu + N(u),$$

where L is some linear differential operator and N is some nonlinear term. Examples include reaction diffusion equations, such as $u_t = \Delta u + N(u)$, or wave equations, such as $u_{tt} - \Delta u = f(u)$, where in the latter case one can write

$$U_t := \frac{\partial}{\partial t}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \Delta & 0 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} 0 \\ f(u) \end{pmatrix} =: LU + N(U).$$

The motivation for first focusing on the linear case, $N = 0$, is as follows. Suppose we begin with a general nonlinear equation $u_t = F(u)$ that has a particular stationary solution of interest, $u(x,t) = u_*(x)$: ie

$F(u_*) = 0$. Formally applying the Ansatz $u = u_* + v$ and expanding the function N in a Taylor series, we obtain

$$v_t = (u_* + v)_t = F(u_* + v) = F(u_*) + DF(u_*)v + \mathcal{O}(v^2) = DF(u_*)v + \mathcal{O}(v^2).$$

When v is small, which corresponds to the original solution u being near the particular solution u_*, the $\mathcal{O}(v^2)$ terms are small when compared with the linear term $DF(u_*)v$. Therefore, a reasonable approximation to the evolution of $u_t = F(u)$ near the solution u_* is

$$v_t = DF(u_*)v =: Lv,$$

which is referred to as the linearization of the PDE at the solution u_*. If solutions to this linear equation remain small (for small initial data), or even decay to zero, then we can hope that in many situations this will imply the same is true (locally) for the full, nonlinear equation. (Although one should be a bit careful – the linearization does not always yield useful information about the nonlinear PDE.)

Therefore, for these four lectures we will focus on linear equations of the form

$$u_t = Lu, \qquad u \in X, \tag{1.1}$$

where L is some linear operator on the Banach space X. For the moment, assume that solutions to this equation exist for all initial conditions $u_0 \in X$. Note that "stability" will generally refer to stability of the zero solution of equation (1.1). This is because we are viewing this linear equation as describing, approximately, the evolution of a perturbation to a particular solution of interest. Thus, if that perturbation stays near zero, then the particular solution is stable.

Definition 1.1. *The solution $u(t) \equiv 0$ of (1.1) is said to be* **stable** *if, given any $\epsilon > 0$, there exists a $\delta = \delta(\epsilon) > 0$ such that, for all $u_0 \in X$ with $\|u_0\|_X \le \delta$, the corresponding solution to (1.1) satisfies $\|u(t)\|_X \le \epsilon$ for all $t \ge 0$. If in addition there exists a δ^* such that for all initial conditions with $\|u_0\|_X < \delta^*$ the corresponding solution satisfies $\lim_{t\to\infty} \|u(t)\|_X = 0$, then the zero solution is said to be* **asymptotically stable**.

Our basic goal will be to construct solutions to equation (1.1) using an object called the semigroup associated with the linear operator L. This is a family of operators, $\{e^{Lt}\}_{t\ge 0}$, that is for each $t \ge 0$ a bounded linear operator on the Banach space X. (Below it will be defined more precisely.) For any initial condition u_0, the solution to (1.1) will be given by $u(t) = e^{Lt}u_0$. This approach is useful because, once the semigroup can be proven to exist, one immediately gets existence and uniqueness of solutions to the linear PDE. Furthermore, in many cases the spectral properties of the operator L can be used to determine the asymptotic behavior of solutions to the linear PDE.

We note that there are many cases where it is better to study the asymptotic behavior of solutions to (1.1) from a different point of view, for example using energy estimates. We do not have time to go into these other methods in detail, although see the remarks in §5.

1.1 Exercises

1.1.1 Standing wave of a reaction diffusion equation

This example comes from [San], which can be found at `http://www.dam.brown.edu/people/sandsted/documents/evans-function-example.pdf`. Consider the nonlinear equation $u_t = u_{xx} - u + u^3$, $u \in \mathbb{R}$,

$x \in \mathbb{R}$, which has an explicit standing pulse given by $u_*(x) = \sqrt{2}\mathrm{sech}(x)$. Show that, near this standing pulse, the PDE can be written $v_t = Lv + N(v)$, where $Lv = v_{xx} + (6\mathrm{sech}^2(x) - 1)v$ and $N(v) = 3u_*v^2 + v^3$. Thus, the linearization of the PDE at this pulse is $v_t = v_{xx} + (6\mathrm{sech}^2(x) - 1)v$.

1.1.2 Traveling wave of the viscous Burgers equation

The viscous Burgers equation is given by $u_t = u_{xx} - uu_x$, and it has an explicit family of traveling waves given by $u_*(x,t) = c - 2\alpha\tanh(\alpha(x - ct))$ for $\alpha, c \in \mathbb{R}$. In terms of the moving coordinate frame $(\xi, t) := (x - ct, t)$, this solution is stationary. Show that, with respect to these coordinates, the dynamics near the traveling wave can be written $v_t = v_{\xi\xi} + cv_\xi - (u_*v)_\xi - vv_\xi$, so $Lv = v_{\xi\xi} + cv_\xi - (u_*v)_\xi$ and $N(v) = -vv_\xi$. More information about the stability of the traveling wave in Burgers equation can be found, for example, in [Zum11].

1.1.3 Sine-Gordon equation

Consider the sine-Gordon equation,

$$u_{tt} - u_{xx} + \sin u = 0.$$

Show that, for $\xi = x - ct$, $|c| < 1$,

$$u_*(\xi) = 4\arctan\left(\pm\exp\left(\frac{\xi}{\sqrt{1-c^2}}\right)\right)$$

is a stationary solution in the moving frame (ξ, t), and that the Ansatz $u = u_* + v$ leads to

$$\frac{d}{dt}\begin{pmatrix} v \\ w \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 \\ (1-c^2)\partial_\xi - \cos(u_*) & 2c\partial_\xi u_* \end{pmatrix}}_{L}\begin{pmatrix} v \\ w \end{pmatrix} + \underbrace{\begin{pmatrix} 0 \\ 2cv_\xi v_t - \sin(u_* + v) + \sin(u_*) + \cos(u_*)v \end{pmatrix}}_{N}.$$

More information about the stability of this solution can be found, for example, in[DDvGV03, JM12].

1.1.4 KdV soliton

The Korteweg–deVries (KdV) equation is

$$u_t + u_{xxx} + uu_x = 0,$$

and it has a solution known as a soliton, or solitary wave, given by $u_*(\xi) = 3c\mathrm{sech}^2(\sqrt{c}\xi/2)$, for $\xi = x - ct$. Show that if $u = u_* + v$, then $v_t = Lv + N(v)$, where $Lv = -v_{\xi\xi\xi} + cv_\xi - (u_*v)_\xi$ and $N(v) = -vv_\xi$. For more information about the stability of this solution, see for example [BSS87].

1.1.5 2D Navier-Stokes

Consider the forced, two-dimensional incompressible Navier-Stokes equation on the torus, written in terms of the vorticity of the fluid ω:

$$\omega_t = \Delta\omega - \mathbf{u} \cdot \nabla\omega + \sin x, \qquad \omega \in \mathbb{R}, \qquad (x, y) \in [-\pi, \pi] \times [\pi, \pi],$$

where $\omega(x-\pi, y) = \omega(x+\pi, y)$ and $\omega(x, y-\pi) = \omega(x, y+\pi)$. The vector $\mathbf{u}$ represents the velocity of the fluid, and, in the case $\int_{\mathbb{T}^2} \omega = 0$ (which is preserved by the flow), the velocity can be written in terms of the vorticity via the Biot-Savart law: $\mathbf{u} = (-\partial_y \Delta^{-1}\omega, \partial_x \Delta^{-1}\omega)$. Due to the forcing function $\sin x$, there exists the stationary solution $\omega_*(x, y) = \sin x$, which is sometimes referred to as shear, or Kolmogorov, flow. Check that this is, in fact, a stationary solution, and show that the linearization around it is $v_t = \Delta v - \partial_y \sin x(1 + \Delta^{-1})v$. For more information, see for example [MS61].

2 Review of linear stability in finite dimensions

Consider the finite-dimensional linear equation

$$u_t = Au, \qquad u \in \mathbb{R}^n, \qquad A \in \mathbb{R}^{n\times n}, \qquad t \in \mathbb{R}. \tag{2.1}$$

We review how to explicitly construct the semigroup associated with (2.1), and how it can be used to determine stability. For more details see, for example, [Chi06]. Recall that, for a matrix A,

$$\|A\| = \sup_{|v|=1} |Av|.$$

Definition 2.1. *Given an $n \times n$ matrix A and $t \in \mathbb{R}$, we define the matrix exponential to be*

$$e^{At} := \sum_{k=0}^{\infty} \frac{(At)^k}{k!}. \tag{2.2}$$

One can check that this sum is well defined by using the fact that $\|A^k\| \le \|A\|^k$ to show that the sum is absolutely convergent (which implies the partial sums form a Cauchy sequence). In fact,

$$\left\| \sum_{k=0}^{n} \frac{(At)^k}{k!} \right\| \le \sum_{k=0}^{\infty} \frac{|t|^k \|A\|^k}{k!} = e^{|t|\|A\|} < \infty.$$

One can use (2.2) to check that

- e^{At} is for each t itself an $n \times n$ matrix
- $e^{At}u_0|_{t=0} = u_0$ and $\frac{d}{dt} e^{At}u_0 = Ae^{At}u_0 = e^{At}Au_0$.
- $e^{(A+B)t} = e^{At}e^{Bt}$ if and only if $AB = BA$.
- $e^{A(t+s)} = e^{At}e^{As}$ for all $t, s \in \mathbb{R}$

The second property implies that $e^{At}u_0$ is the unique solution to (2.1) with initial condition u_0. The final property is referred to as the group property for the following reason. The exponential function $e^{A\cdot}$ is a map from the group of real numbers with the operation of addition to the group of invertible $n \times n$ matrices with the operation of multiplication. The exponential function preserves the group operations through the relationship $e^{A(t+s)} = e^{At}e^{As}$. Below we will encounter a situation where a similar property holds, but only for $t, s > 0$. It will then be called the semi-group property, as the inverse e^{-At} will not necessarily be well defined. For this reason, the family $\{e^{At}\}$ is often referred to as the semi-group generated by the linear operator A.

If A is diagonalizable, then $A = SDS^{-1}$ for some nonsingular matrix S, where D is a diagonal matrix whose entries $\{\lambda_k\}_{k=1}^n$ are exactly the eigenvalues of A. One then has $A^k = (SDS^{-1})^k = SD^kS^{-1}$, and so by (2.2) we have $e^{At} = Se^{Dt}S^{-1}$. By taking powers of the diagonal matrix D, one finds

$$e^{At} = S\sum_{k=0}^{\infty}\frac{(Dt)^k}{k!}S^{-1} = S\begin{pmatrix} e^{\lambda_1 t} & 0 & \dots & 0 \\ 0 & e^{\lambda_2 t} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & e^{\lambda_n t} \end{pmatrix} S^{-1}. \tag{2.3}$$

One can further characterize the matrix S in terms for the eigenvectors associated with A. If A is not diagonalizable, a similar formula to the one above is possible, using the Jordan normal form of A, although it is slightly more complicated. Using (2.3) (or its generalization in the non-diagonalizable case), one can prove the following result.

Proposition 2.2. *The zero solution of (2.1) is stable if and only if $Re\lambda_k \leq 0$ for all k and any eigenvalue with zero real part has its geometric multiplicity equal to its algebraic multiplicity. The zero solution is asymptotically stable if and only $Re\lambda_k < 0$ for all k.*

Therefore, in finite dimensions, stability can be characterized by the eigenvalues of the matrix A, via the semigroup e^{At}.

3 Linear stability in infinite dimensions

We'd like to understand to what extend the above finite-dimensional result is also true in infinite dimensions: can we solve the equation $u_t = Lu$ with an object $e^{Lt}u_0$ and connect the asymptotic behavior of solutions with the spectral properties of the operator L? To begin, we first review some notions from the spectral theory for linear operators on a Banach space. Much of this material is taken from [EN00], where more details can be found.

3.1 Spectral theory in infinite dimensions

Consider a linear operator L on a Banach space X. In general, one must be careful to specify the domain of the operaror L, which will be denoted by $D(L) \subset X$. Thus, $L : D(L) \subset X \to X$. Much of the theory also requires that the operator be closed (if $\{u_k\} \subset D(L)$, $u_k \to u$ in X, and $Lu_k \to v$ in X, then $u \in D(L)$ and $Lu = v$) and densely defined ($D(L)$ is dense in X), which holds in most reasonable settings. We'll try to avoid unnecessary technicalities associated with these issues.

A good example to keep in mind throughout this section is the Laplacian, $L = \Delta$, posed on the Banach space $X = L^2(\mathbb{R}^n)$. One can check that with $D(\Delta) = H^2(\mathbb{R}^n)$ this operator is closed and densely defined. For details, we refer to [Mik98]. Many of the results that apply to the Laplacian also apply to operators of the form $L = \triangle + B$, where B is sufficiently nice, as we'll see in §4 below.

Definition 3.1. *Given a linear operator L on a Banach space X, the spectrum of L, denoted by $\Sigma(L)$, is the set of all $\lambda \in \mathbb{C}$ such that the operator $(\lambda - L)$ does not have a bounded inverse defined on all of X. The resolvent set is $\rho(L) = \mathbb{C} \setminus \Sigma(L)$. Equivalently, $\rho(L)$ is the set of all λ such that $(\lambda - L) : D(L) \to X$ is bijective. On the resolvent set, we define the resolvent operator to be $(\lambda - L)^{-1}$.*

Definition 3.2. *A complex number λ is called an eigenvalue of L if the operator $(\lambda - L)$ has a nontrivial null space in X. In other words, there exists a $u \in X$, $u \neq 0$, such that $Lu = \lambda u$. Equivalently, $(\lambda - L)$ is not injective.*

In finite dimensions, the only one way for $(\lambda - L)$ to fail to have a bounded inverse on all of $\mathbb{R}^n$ is if λ is an eigenvalue. Hence, the spectrum of an matrix is exactly the set of its eigenvalues. In infinite dimensions, however, there are more ways for $(\lambda - L)$ to fail to have a bounded inverse. For example, its range could fail to be dense and/or fail to be closed. One can check (see [EN00, Lemma 1.9]) that the range not being closed is equivalent to there existing a sequence of so-called approximate eigenvalues: $(\lambda - L)u_k \to 0$ as $k \to 0$, where $u_k \in D(L)$ for all k, $\|u_k\| = 1$, but the sequence $\{u_k\}$ does not have a limit in X.

As an example, consider the Laplacian in one dimension: $L = \partial_x^2$, $X = L^2(\mathbb{R})$. For a given λ, we can try to solve the equation $(\lambda - L)v = w$ for a given $w \in X$ via $v = (\lambda - L)^{-1}w$. Denoting the Fourier transform by $\mathcal{F}(v) = \hat{v}$, we find

$$v = (\lambda - L)^{-1}w, \qquad \hat{v}(k) = \frac{1}{k^2 + \lambda}\hat{w}, \qquad v(x) = \mathcal{F}^{-1}\left(\frac{1}{(\cdot)^2 + \lambda}\hat{w}(\cdot)\right)(x).$$

Since $k \in \mathbb{R}$, for any $\lambda \in (-\infty, 0]$ one can find a $w \in L^2(\mathbb{R})$ such that $\|v\|_{L^2}$ is not finite. Hence, $(\lambda - L)^{-1}$ is not well defined on all of $L^2(\mathbb{R})$ for any such λ. In addition, one can prove using the above formulation that for $\lambda \notin (-\infty, 0]$ the resolvent operator is well-defined and bounded on all of $L^2(\mathbb{R})$. Hence, $\Sigma(\partial_x^2) = (-\infty, 0]$. On the other hand, if we fix $\lambda \in (-\infty, 0]$ and look for eigenvalues, we find $u_{xx} = \lambda u$. This implies $u(x; \lambda) = e^{\sqrt{\lambda}x} = e^{\mathrm{i}\sqrt{|\lambda|}x}$, which is not in $L^2(\mathbb{R})$. One can check that for each such λ there is a sequence u_k of approximate eigenvalues. For example, one can use mollifiers of the form $e^{-\epsilon x^2}$ and set $u_k(x; \lambda) = C_k e^{\mathrm{i}\sqrt{|\lambda|}x} e^{-\epsilon_k x^2}$, where $\epsilon_k \to 0$ as $k \to \infty$ and the constant C_k is chosen so that $\|u_k\|_X = 1$.

Because there are multiple ways for the resolvent operator to fail to be bounded on all of X, it is useful to define subsets of the spectrum in terms of the way(s) in which $(\lambda - L)$ fails to have a bounded inverse.

Definition 3.3. *An operator is said to be Fredholm if its range is closed and both the dimension of its null space and the co-dimension of its range are finite. The Fredholm index of such an operator is the dimension of its null space minus the co-dimension of its range.*

Definition 3.4. *[San02] The point spectrum, Σ_{pt}, is the set of all elements of the spectrum for which $(\lambda - L)$ is Fredholm with index zero. The essential spectrum is defined to be $\Sigma(L)_{\mathrm{ess}} = \Sigma(L) \setminus \Sigma_{\mathrm{pt}}(L)$.*

One needs to be slightly careful with this definition. In this formulation, the point spectrum is not equivalent to the set of eigenvalues. The reason is that eigenvalues can be embedded in the essential spectrum. This will be reflected below in example 3.1.2. (Some texts, including [EN00], define the point spectrum to be the set of all eigenvalues.)

Definition 3.5. *[EN00] The approximate point spectrum is the set of all λ such that $(\lambda - L)$ has a range which is not closed in X. It will be denoted by Σ_{app}, and we note $\Sigma_{app} \subset \Sigma_{ess}$.*

Dividing the spectrum into the point and essential spectrum is useful when computing the spectrum, and the approximate point spectrum is where the Spectral Mapping Theorem (stated below) can fail.

Definition 3.6. *The residual spectrum, denoted $\Sigma_{\rm res}(L)$, is the set of all λ for which the range of $(\lambda - L)$ is not dense in X.*

The residual spectrum can sometimes be computed using the fact that it coincides with the set of all eigenvalues of the Banach space adjoint of L [EN00, §IV.1 Prop 1.12].

3.1.1 Example: Standing wave of a reaction diffusion equation

This is a continuation of the example in 1.1.1. We'll work in $X = L^2(\mathbb{R})$. The equation $\lambda u = Lu$ can be written

$$\lambda u = u_{xx} + (6\text{sech}^2(x) - 1)u.$$

Since this is a second-order ODE, we know that for each value of λ there are two independent solutions. One can check that

$$\begin{aligned} u_1(x;\lambda) &= e^{\sqrt{1+\lambda}x}\left[1 + \frac{\lambda}{3} - \sqrt{1+\lambda}\tanh(x) - \text{sech}^2(x)\right] \\ u_2(x;\lambda) &= e^{-\sqrt{1+\lambda}x}\left[1 + \frac{\lambda}{3} + \sqrt{1+\lambda}\tanh(x) - \text{sech}^2(x)\right] \end{aligned}$$

are indeed two independent solutions of the above equation. (They can be found using hypergeometric series.) In order to investigate the spectrum, we need to determine when the resolvent operator is well defined. It turns out that, for this example, we can calculate it explicitly as follows. Suppose we are given a function w and we seek a function u such that $(\lambda - L)u = w$; hence, $u = (\lambda - L)^{-1}w$. We'll use the method of variation of parameters, which means that we assume the function u has the form

$$u(x;\lambda) = v_1(x;\lambda)u_1(x;\lambda) + v_2(x;\lambda)u_2(x;\lambda),$$

and solve for the functions $v_{1,2}$ in terms of w. To do this, we impose the condition that $v_1'u_1 + v_2'u_2 = 0$, which is one equation, and insert the above form of u into the equation $(\lambda - L)u = w$ to obtain a second equation. These two equations can be written

$$\begin{pmatrix} v_1' \\ v_2' \end{pmatrix} = \frac{1}{u_1u_2' - u_2u_1'}\begin{pmatrix} u_2' & -u_2 \\ -u_1' & u_1 \end{pmatrix}\begin{pmatrix} 0 \\ -w \end{pmatrix} = -\frac{9}{2\lambda\sqrt{1+\lambda}(3-\lambda)}\begin{pmatrix} u_2' & -u_2 \\ -u_1' & u_1 \end{pmatrix}\begin{pmatrix} 0 \\ -w \end{pmatrix},$$

where to obtain the final equality we have used the above expressions for $u_{1,2}$ to explicitly calculate the Wronskian $u_1u_2' - u_2u_1'$. We can see immediately that there will be problems if $\lambda \in \{0,3\} \cup (-\infty,-1]$. Continuing our calculation of $v_{1,2}$ (for λ not in this bad set), we find

$$v_1' = -\frac{9}{2\lambda\sqrt{1+\lambda}(3-\lambda)}u_2w, \qquad v_2' = \frac{9}{2\lambda\sqrt{1+\lambda}(3-\lambda)}u_1w.$$

To integrate these expressions, we note that u_1 is well-behaved at $-\infty$ while u_2 is well-behaved at $+\infty$. Hence, we define

$$v_1(x;\lambda) = \frac{9}{2\lambda\sqrt{1+\lambda}(3-\lambda)}\int_x^\infty u_2(y;\lambda)w(y)\mathrm{d}y, \qquad v_2(x;\lambda) = \frac{9}{2\lambda\sqrt{1+\lambda}(3-\lambda)}\int_{-\infty}^x u_1(y;\lambda)w(y)\mathrm{d}y.$$

Inserting these formulas back into the expression for u, one finds that the solution can be written

$$\begin{aligned} u(x) &= \frac{9}{2\lambda\sqrt{1+\lambda}(3-\lambda)} \int_{\mathbb{R}} [u_1(x;\lambda)u_2(y;\lambda)H(y-x) + u_2(x;\lambda)u_1(y;\lambda)H(x-y)]w(y)\mathrm{d}y \\ &=: \int_{\mathbb{R}} G(x,y;\lambda)w(y)\mathrm{d}y. \end{aligned}$$

Therefore, the action of the resolvent operator can be expressed through the integral kernel $G(x,y;\lambda)$. One can now investigate for which values of λ this operator is well-defined and bounded on all of L^2 and prove that $\Sigma_{\mathrm{pt}}(L) = \{0,3\}$ and $\Sigma_{\mathrm{ess}}(L) = \Sigma_{\mathrm{app}}(L) = (-\infty,-1]$. More precisely, if $\lambda \in (-\infty,-1]$, then the functions $u_{1,2}$ are bounded and can be used to construct approximate eigenvalues as we did above for the Laplacian. If $\lambda = 0$ or $\lambda = 3$, then the kernel of $(\lambda - L)$ is one-dimensional: $-u_1(x;\lambda) = u_2(x;0) = \mathrm{sech}(x)\mathrm{tanh}(x)$ and $u_1(x;3) = u_2(x;3) = \mathrm{sech}^2(x)$, which are the corresponding eigenfunctions. (In each case, the second, linearly independent solution to $\lambda u = u_{xx}$ is a function that's not in L^2.) Let ϕ denote the eigenfunction in either case. Since L^2 is a Hilbert space, we can set $E = \mathrm{span}\{\phi\}$ and let $E^\perp$ denote its orthogonal complement, which is closed. Since L is self-adjoint, $E^\perp$ is the range of $(\lambda - L)$ (see, for example, [Bre11, §2.6 Cor 2.18]), and we see that the Fredholm index is zero. For all other values of λ, the explicit form of G can be used to show that the above resolvent operator is bounded on all of X.

3.1.2 Exercise: Traveling wave of the viscous Burgers equation

This is a continuation of the example in 1.1.2. As in the previous example we will work in $L^2(\mathbb{R})$, and one can find two independent solutions to the eigenvalue equation $(L-\lambda)v = 0$, which are

$$\begin{aligned} v_1(x;\lambda) &= \mathrm{sech}(\alpha\xi)(\sqrt{\alpha^2+\lambda} - \alpha\mathrm{tanh}(\alpha\xi))e^{\sqrt{\alpha^2+\lambda}\xi} \\ v_2(x;\lambda) &= \mathrm{sech}(\alpha\xi)(-\sqrt{\alpha^2+\lambda} - \alpha\mathrm{tanh}(\alpha\xi))e^{-\sqrt{\alpha^2+\lambda}\xi}. \end{aligned}$$

These solutions can be derived by noting that if $V(x) = \int_{-\infty}^x v(y)\mathrm{d}y$ and $V(x) = \mathrm{sech}(\alpha\xi)u(x)$, then $u_{xx} - (\alpha^2+\lambda)u = 0$. Use the method of variation of parameters to show that the kernel of the resolvent operator is given by

$$\begin{aligned} v(\xi) &= \frac{1}{2\lambda\sqrt{\alpha^2+\lambda}} \int_{\mathbb{R}} \mathrm{sech}(\alpha\xi)\mathrm{cosh}(\alpha y)g(\xi,y)e^{-\sqrt{\alpha^2+\lambda}(y-x)}H(y-x)w(y)\mathrm{d}y \\ &\quad + \frac{1}{2\lambda\sqrt{\alpha^2+\lambda}} \int_{\mathbb{R}} \mathrm{sech}(\alpha\xi)\mathrm{cosh}(\alpha y)g(y,\xi)e^{-\sqrt{\alpha^2+\lambda}(x-y)}H(x-y)w(y)\mathrm{d}y \\ &=: \int_{\mathbb{R}} G(\xi,y;\lambda)w(y)\mathrm{d}y, \end{aligned}$$

where $g(\xi,y) = (\sqrt{\alpha^2+\lambda} + \alpha\mathrm{tanh}(\alpha y))(\sqrt{\alpha^2+\lambda} - \alpha\mathrm{tanh}(\alpha\xi))$.

One can see explicitly from the formulas for $v_{1,2}$ that they are both L^2 functions as long as $\lambda \in \{\lambda \in \mathbb{C} : |\mathrm{Re}\sqrt{\lambda+\alpha^2}| < \alpha\} = \{\lambda \in \mathbb{C} : \mathrm{Re}\lambda < -\mathrm{Im}\lambda/(4\alpha^2)\}$. Therefore, the region in the complex plane strictly to the left of the parabola $\mathrm{Re}\lambda = -\mathrm{Im}\lambda/(4\alpha^2)$ is filled with eigenvalues. However, it is not true that this entire region is contained in $\Sigma_{\mathrm{pt}}(L)$. In fact, we claim that

$$\Sigma_{\mathrm{pt}} = \emptyset, \qquad \Sigma_{\mathrm{app}} = \{\mathrm{Re}\lambda = -\mathrm{Im}\lambda/(4\alpha^2)\}, \qquad \Sigma_{\mathrm{ess}} = \{\mathrm{Re}\lambda < -\mathrm{Im}\lambda/(4\alpha^2)\} \cup \Sigma_{\mathrm{app}}.$$

Prove this using the following facts. By [EN00, Prop1.12, pg 243], the range of $(\lambda - L)$ is not dense in X if and only if λ is an eigenvalue of the adjoint operator, L^*. In this case, $L^* = \partial_\xi^2 - 2\alpha\tanh(\alpha\xi)\partial_\xi$, and one can again explicitly compute the solutions to the equation $(\lambda - L)u = 0$. They are

$$\begin{aligned} u_1^*(\xi, \lambda) &= \left(-\alpha\sinh(\alpha\xi) + \sqrt{\alpha^2+\lambda}\cosh(\alpha\xi)\right) e^{\sqrt{\alpha^2+\lambda}\xi} \\ u_2^*(\xi, \lambda) &= \left(-\alpha\sinh(\alpha\xi) - \sqrt{\alpha^2+\lambda}\cosh(\alpha\xi)\right) e^{-\sqrt{\alpha^2+\lambda}\xi}. \end{aligned}$$

3.2 Semigroup theory

We now turn our attention to the definition of the object e^{Lt} and how it can be used to investigate stability. The majority of these results can be found in [EN00]. Other frequently cited references on semigroup theory are [Paz83] and [CL99]. Also, [Mik98] has a nice introduction, specifically in the context of PDE.

Consider again the equation

$$u_t = Lu, \qquad u \in X, \tag{3.1}$$

where L is some linear operator on the Banach space X. When L is unbounded, it's not possible to define the semigroup as in (2.2) – it's unlikely that the series will converge. However, in many cases one can still construct a family of bounded operators, called a semigroup, that shares many of the properties of the matrix exponential e^{At}.

Definition 3.7. *A family of bounded operators $T(t)$, $t \geq 0$, on a Banach space X is called a strongly-continuous semigroup if $T(0) = I$, $T(t+s) = T(t)T(s)$ for all $t, s \geq 0$, and if it is strongly continuous, meaning that if $t_k \to t$, then for any $u \in X$ we have* $\lim_{k\to\infty} \|T(t_k)u - T(t)u\|_X = 0$.

Remark 3.8. *The term strongly continuous is used because this type of continuity is exactly continuity with respect to what's called the strong operator topology. It is weaker than requiring continuity with respect to the uniform (or operator) topology, which would require*

$$\|T(t_k) - T(t)\| = \sup_{\|u\|_X=1} \|T(t_k) - T(t)u\|_X \to 0.$$

In general, this uniform continuity is not satisfied.

Proposition 3.9. *[EN00, Prop 5.5, §I.5] If $T(t)$ is a strongly continuous semigroup then there exist constants $\eta \in \mathbb{R}$ and $M(\eta) \geq 1$such that $\|T(t)\| \leq Me^{\eta t}$ for all $t \geq 0$.*

This proof is relatively short, and a good way to get used to working with semigroups.

Proof. First, we claim that there exists a $\delta > 0$ and an $M > 0$ such that $\|T(t)\| \leq M$ for all $0 \leq t \leq \delta$. BWOC, if not then there exists a sequence $t_k \to 0$ such that $\|T(t_k)\| \to \infty$. By the uniform boundedness principle ($\sup_k \|T(t_k)x\| < \infty$ for all x implies that $\sup_k \|T(t_k)\| < \infty$), this implies there must be some x such that $\|T(t_k)x\| \to \infty$. But by the strong continuity of the semigroup, $\|T(t_k)x\| \to 0$.

Next, fix this δ and M (taking $M \geq 1$ if it isn't already). For any $t \geq 0$, write $t = s + \delta n$, for $s \in [0, \delta]$. Then we have

$$\|T(t)\| = \|T(s+\delta n)\| = \|T(s)T(\delta)\dots T(\delta)\| \leq \|T(s)\|\|T(\delta)\|^n \leq M^{n+1} = Me^{n\log M} \leq Me^{\eta t},$$

for $\eta := (\log M)/\delta$, and this holds for all $t \geq 0$. ∎

Therefore, if we can associate a strongly continuous semigroup with solutions to equation (3.1), solutions can't blow up in finite time. The constant η will be connected with stability.

Definition 3.10. *The generator L of the strongly continuous semigroup $T(t)$ on the Banach space X is the operator defined by*

$$Lu := \lim_{h \to 0} \frac{T(h)u - u}{h},$$

and the domain of L is all $u \in X$ for which the above limit exists.

One often uses the notation $T(t) = e^{Lt}$ for the semigroup with generator L. Note that this is just notation, and it does not imply that the formula (2.2), with A replaced by L, is well-defined. One can show (almost directly from the above definition) that, in fact,

$$\frac{d}{dt} e^{Lt} u = L e^{Lt} u = e^{Lt} L u,$$

for all $t \geq 0$. Thus, if $T(t)$ is a semigroup with generator L, then $e^{Lt} u_0$ is the unique classical solution of the linear equation (3.1) with initial condition $u_0 \in D(L)$. Conversely, when solving such a PDE, one would like to know under what conditions the operator L is the generator of a strongly continuous semigroup.

Proposition 3.11. *[EN00, Corollary 3.6 and Theorem 3.8, §II.3] An operator L on a Banach space X generates a strongly continuous semigroup satsifying $\|T(t)\| \leq Me^{\eta t}$ if and only if L is closed, densely defined, and every λ with $Re\lambda > \eta$ is in the resolvent set, with the estimate*

$$\|(\lambda - L)^{-n}\| \leq \frac{M}{|\lambda - \eta|^n}$$

holding for all $n \in \mathbb{N}$. If $M = 1$, this last estimate can be replaced by $\|(\lambda - L)^{-1}\| \leq 1/(|\lambda - \eta|)$ – ie if $M = 1$ then one need only check the estimate for $n = 1$.

These results can be summarized as follows. If one has a strongly continuous semigroup, then it must satisfy a growth estimate of the form $\|T(t)\| \leq Me^{\eta t}$ and one can define an associated operator L, called the generator. On the other hand, if one starts with a linear operator L satisfying certain conditions, then there is an associated strongly continuous semigroup. One can prove that the generator uniquely determines the semigroup and vice versa.

Remark 3.12. *It is possible for unique classical ($u(t) \in D(L)$ for all $t \geq 0$) solutions to the linear equation (3.1) to exist without the operator L being the generator of a strongly continuous semigroup. However, in that case there is a closely related operator that is a generator of a strongly continuous semigroup [EN00, Prop 6.6, §II.6]. In general, $T(t)u_0$ is a classical solution for $t \geq 0$ only if $u_0 \in D(L)$. However, if one considers mild solutions, which are solutions of the form*

$$u(t) = L \int_0^t u(s) \mathrm{d}s + u_0,$$

then the semigroup can be used to construct solutions for all $u_0 \in X$.

We now focus on the constant η in Proposition 3.9 and how it is connected with stability.

Definition 3.13. *The growth bound of a strongly continuous semigroup is the quantity*

$$\eta_0(L) = \inf\{\eta : \exists M(\eta) < \infty \text{ such that } \|T(t)\| \leq M(\eta)e^{\eta t} \ \forall \ t \geq 0\}.$$

In finite dimensions, we saw that η_0 was directly connected with the spectrum of L. In infinite dimensions, the relationship is a bit more complex.

Definition 3.14. *The spectral bound of a linear operator L is*

$$s(L) = \sup\{Re\lambda : \lambda \in \Sigma(L)\}.$$

One can prove that $-\infty \leq s(L) \leq \eta_0(L) < \infty$. However, it is possible that $s(L) < \eta_0(L)$, as the following example, taken from [CL99, Example 2.24], shows.

Consider the operator $L = x\partial_x$. One can check that the solution to the equation $u_t = Lu$ with initial condition u_0 is $T(t)u_0 = u_0(e^t x)$. In fact, one can obtain an explicit estimate for the growth bound of this semigroup on L^2:

$$\|T(t)u_0\|^2_{L^2(1,\infty)} = \int_x^\infty |u_0(e^t x)|^2 \mathrm{d}x = e^{-t}\int_{e^t}^\infty |u_0(y)|^2 \mathrm{d}y \leq e^{-t}\|u_0\|^2_{L^2(1,\infty)}.$$

Since $s(L) \leq \eta_0(L)$, this implies that $\Sigma(L) \subset \{\mathrm{Re}(\lambda) \leq -1/2\}$. This in turn implies $\{\mathrm{Re}(\lambda) > -1/2\} \subset \rho(L)$, and so for any such λ, there is a $C > 0$ for which $\|(\lambda - L)^{-1} f\|_{L^2(1,\infty)} \leq C\|f\|_{L^2(1,\infty)}$ for all $f \in L^2(1,\infty)$.

We now claim that on the space $X = H^1(1,\infty)$, $s(L) = -1/2 < 1/2 \leq \eta_0(L)$. One can check that $T(t)$ is still a strongly continuous semigroup on this space and solve the eigenvalue equation $\lambda u = Lu$ explicitly to find $u(x;\lambda) = Cx^\lambda$. This is in H^1, and hence an eigenfunction, whenever $\mathrm{Re}\lambda < -1/2$. The resolvent operator can also be computed explicitly. If $(\lambda - L)u = v$, then

$$u(x) = [(\lambda - L)^{-1}v](x) = -x^\lambda \int_x^\infty y^{-(\lambda+1)} v(y)\mathrm{d}y. \tag{3.2}$$

By the above argument in L^2, we know that $\|u\|_{L^2} = \|(\lambda - L)^{-1}v\|_{L^2} \leq C\|v\|_{L^2} \leq C\|v\|_{H^1}$. In addition, if $\mathrm{Re}\lambda > -1/2$, a direct estimate on the above formula can be used to show that $\|u_x\|^2_{L^2} \leq C\|v\|^2_{H^1}$. Thus, the resolvent operator is bounded on H^1 for all such λ, which implies $s(L) = -1/2$. To see the growth bound is positive, fix $t \geq 0$ and pick $u_0 \in H^1(1,\infty)$ such that $\mathrm{supp} u_0 \subset (e^t, \infty)$ and $\|\partial_x u_0\|_{L^2(1,\infty)} = 1$. We then have

$$\|T(t)u_0\|^2_{H^1} \geq \|\partial_x T(t)u_0\|^2_{L^2} = e^t \int_{e^t}^\infty |u_0'(y)|^2 \mathrm{d}y = e^t.$$

Hence, if η is such that $\|T(t)\|_{\mathcal{L}(H^1)} \leq Me^{\eta t}$ for all $t \geq 0$, we must have $\eta \geq 1/2$.

Therefore, in infinite dimensions, one must be careful about using only spectral information to determine stability. Luckily, for a large class of operators, one can show that $s(L) = \eta_0(L)$. This is done using so-called spectral mapping theorems. In preparation for stating this theorem, we first consider the following proposition.

Proposition 3.15. *[EN00, §IV.2, Prop 2.2] For any fixed $t \geq 0$, define the spectral radius of the strongly continuous semigroup $T(t)$ at time t to be $r(T(t)) = \sup\{|\lambda| : \lambda \in \Sigma(T(t))\}$. One has*

$$\eta_0 = \frac{1}{t_0}\log(r(T(t_0))) \tag{3.3}$$

for each $t_0 > 0$. In particular, $r(T(t)) = e^{\eta_0 t}$.

Proof. We only outline the proof here, and refer to [EN00] for details. The first step is to show that

$$\eta_0 = \inf_{t>0} \frac{1}{t} \log \|T(t)\| = \lim_{t\to\infty} \frac{1}{t} \log \|T(t)\|.$$

The first equality is perhaps believable from the definition of the growth bound, and the second equality can be proven for a general class of subadditive ($f(t+s) \le f(t) + f(s)$) functions that are bounded on compact intervals; $\log \|T(t)\|$ falls into this category. To prove the right hand side of the above equation is in fact equal to the right hand side of (3.3), one can use the Hadamard formula for the spectral radius of an operator:

$$r(T(t)) \underbrace{=}_{\text{Hadamard}} \lim_{n\to\infty} \|T(t)^n\|^{1/n} = \lim_{n\to\infty} \|T(nt)\|^{1/n} = \lim_{n\to\infty} e^{\frac{t}{nt}\log\|T(nt)\|} = e^{t \lim_{n\to\infty} \frac{1}{nt}\log\|T(nt)\|} = e^{\eta_0 t}.$$

∎

Therefore, we can determine the growth bound of the semigroup, and hence stability, by determining its spectral radius. Essentially, one would like to be able to show that

$$\Sigma(T(t)) \setminus \{0\} = e^{\Sigma(L)t} = \{e^{\lambda t} : \lambda \in \Sigma(L)\}. \tag{3.4}$$

This would imply that, if $\mathrm{Re}(\lambda) \le -\delta < 0$ for all $\lambda \in \Sigma(L)$, then $e^{\eta_0 t} = r(T(t)) = e^{-\delta t}$ for all t, and so $\eta_0 = -\delta$. Hence, we'd have asymptotic stability. The problem is that, as we saw in the above example, (3.4) does not hold for a general strongly continuous semigroup. The problem has to do with the approximate point spectrum.

Proposition 3.16. *[EN00, §IV.3] For any strongly continuous semigroup e^{Lt} on a Banach space with generator L, $e^{\Sigma(L)t} \subset \Sigma(e^{Lt})$. However, the reverse inclusion (without zero), $\Sigma(e^{Lt}) \setminus \{0\} \subset e^{\Sigma(L)t}$, does not necessarily hold on the approximate point spectrum. In particular, it may be possible to find $\mu \in \Sigma_{\mathrm{app}}(e^{Lt}) \setminus \{0\}$ such that $\mu \neq e^{\lambda t}$ for all $\lambda \in \Sigma(L)$.*

Luckily, there are ways to prove that the spectral mapping theorem does hold on the entire spectrum for certain (classes of) operators. We mention two ways to do this. First, recall that $T(t)$ was defined to be a strongly continuous semigroup based upon the type of continuity one had at $t = 0$. It turns out there is a small class of semigroups which have even nicer behavior in terms of smoothness. We do not have time to go into details here, but we mention one particular class, analytic semigroups, which arises in applications. Analytic semigroups are analytic in the parameter t, at least in some region of the complex plane containing $\mathbb{R}^+$. They can be characterized by a resolvent bound and the location of the spectrum of their generator, and the spectral mapping theorem can be proven to hold for this class. More precisely, we have the following theorem.

Theorem 1. *[EN00, §IV.3, Cor 3.12] Let L be a closed and densely defined linear operator on a Banach space X. Suppose further that there exists a $\delta > 0$ such that the sector*

$$S_{\pi/2+\delta} = \left\{\lambda \in \mathbb{C} : |arg\lambda| < \frac{\pi}{2} + \delta\right\} \setminus \{0\}$$

is contained in the resolvent set of L, and for all $\epsilon \in (0,\delta)$ there exists an $M_\epsilon \ge 1$ such that

$$\|(\lambda - L)^{-1}\| \le \frac{M_\epsilon}{|\lambda|}, \qquad \forall \lambda \in \bar{S}_{\pi/2+\delta-\epsilon} \setminus \{0\}.$$

Then the semigroup associated with L satisfies the spectral mapping theorem (3.4).

Such an operator L is called sectorial, and it generates an analytic semigroup. In fact, the only operators that generate analytic semigroups are sectorial ones. (There is, however, a slightly larger class of semigroups, effectively between analytic and strongly continuous semigroups, for which the spectral mapping theorem also holds. See [EN00] for more details.)

If one is lucky enough to work with a PDE where the linear operator is sectorial, then determining the spectrum of the operator is sufficient for proving linear stability.

Remark 3.17. *One must a little bit careful when the growth bound is zero. For example, even if the spectral mapping theorem holds, it is not necessarily true that Re*$(\lambda) \leq 0$ *for all* $\lambda \in \Sigma(L)$ *implies stability. It is true that it implies* $e^{\eta_0 t} = r(T(t)) \leq 1$ *for all* t*, and so* $\eta_0 = 0$*. However, there could be a solution that grows algebraically:* $\sup_{\|u_0\|=1} \|T(t)u_0\| = C(t+1)$ *for some* C*. In this case, one can check that for each* $\epsilon > 0$ *there is an* $M_\epsilon = \mathcal{O}(1/\epsilon)$ *such that* $\|T(t)\| \leq M_\epsilon e^{\epsilon t}$*. Thus,* $\eta_0 = \inf\{\epsilon > 0\} = 0$*, despite the fact that zero is not stable.*

If the operator is not sectorial and X is in fact a Hilbert space, then the following result, often referred to as the Gearhart or Gearhart-Prüss Theorem, may be useful.

Theorem 2. *[CL99, Thm 2.10] Let* L *be the generator of a strongly continuous semigroup* $T(t)$ *on a Hilbert space* H*. The set* $\Sigma(T(t)) \setminus \{0\}$ *is the set of all complex numbers of the form* $e^{\lambda t}$ *where either* $\lambda + 2\pi \mathrm{i}k/t \in \Sigma(L)$ *for some* $k \in \mathbb{Z}$ *or the sequence* $\{\|(\lambda + 2\pi \mathrm{i}k/t - L)\|\}_{k\in\mathbb{Z}}$ *is unbounded.*

Because of the fact that the resolvent set is open and the function $\lambda \to (\lambda - L)^{-1}$ is locally analytic, this theorem implies that the growth bound of a strongly continuous semigroup on a Hilbert space is strictly negative if and only if both $\{\lambda : \mathrm{Re}\lambda \geq 0\} \subset \rho(L)$ and there exists a $C \geq 0$ such that $\|(ik - L)^{-1}\| \leq C$ for all $k \in \mathbb{R}$.

In the next section, we illustrate how the above ideas can be used to prove that the spectral mapping theorem does hold for a large class of PDE – namely reaction-diffusion equations with sufficiently nice coefficients.

3.3 Exercises

The purpose of these exercises is to fill in some of the details missing from the above example where $T(t)u_0 = u_0(e^t x)$ and the generator is $L = x\partial_x$

Exercise 1 Show that the semigroup on $L^2(1,\infty)$ is indeed strongly continuous using the following steps.

- Show that if $\lim_{t\to 0} \|T(t)u_0 - u_0\|_{L^2(1,\infty)} = 0$ then $\lim_{t_k \to t} \|T(t_k)u_0 - T(t)u_0\|_{L^2(1,\infty)} = 0$ for all sequences $t_k \to t$.
- Prove that $\lim_{t\to 0} \|T(t)\phi - \phi\|_{L^2(1,\infty)} = 0$ for $\phi \in C^\infty(1,\infty) \cap L^2(1,\infty)$. Hint: separate the interval of integration into $(1,R)$ and (R,∞), and use the fact that $\phi(e^t x) - \phi(x) = \int_x^{e^t x} \phi'(y)\mathrm{d}y$.
- Use a density argument (smooth functions are dense in L^2) to extend this result to all of L^2.

Exercise 2 Show, using (3.2), that if $\text{Re}\lambda > -1/2$ then $\|u_x\|^2_{L^2} \leq C\|v\|^2_{H^1}$.

4 Spectral mapping theorem for reaction-diffusion equations

In this section we consider the class of equations

$$u_t = \partial_x^2 u + b(x)\partial_x u + c(x)u =: Lu, \tag{4.1}$$

where $u \in \mathbb{R}$, $x \in \mathbb{R}$, and $b, c : \mathbb{R} \to \mathbb{R}$ are smooth and bounded. We'll show that the spectral mapping theorem holds for the semigroup generated by L on the space $L^2(\mathbb{R})$, and hence stability can be proven simply by determining the spectrum of L. We note that this result can be generalized slightly, for example by extending to $\mathbb{R}^n$, by weakening the regularity conditions on b and c, or by replacing the Laplacian by $\nabla \cdot (A(x)\nabla u)$, where A is sufficiently smooth and this operator is uniformly elliptic.

The idea is to first show that the Laplacian is sectorial, and hence generates an analytic semigroup. We'll then appeal to a perturbation result, which implies that L must also generate an analytic semigroup. Finally, we recall that spectral mapping theorem holds whenever the generator is sectorial, which is the case for the generator of an analytic semigroup.

4.1 The Laplacian

Consider the operator ∂_x^2. We've already seen above that $\Sigma(\partial_x^2) = (-\infty, 0]$, and this certainly lies in a sector. Therefore, we just need to obtain the bound on the resolvent operator. By solving the equation $(\lambda - \partial_x^2)u = v$ using the method of variation of parameters, as in the examples in §3.1.1 and §3.1.2, above, one finds that

$$u(x) = \int G(x, y; \lambda)v(y)\text{d}y, \qquad G(x, y, \lambda) = \frac{1}{2\sqrt{\lambda}}\left(H(x-y)e^{-\sqrt{\lambda}(x-y)} + H(y-x)e^{-\sqrt{\lambda}(y-x)}\right).$$

Therefore, the above integral is just a convolution: $u = G * v$. In general, one has $\|f * g\|_{L^p} \leq C\|f\|_{L^q}\|g\|_{L^r}$ when $1/p + 1 = 1/q + 1/r$. Taking $p = r = 2$ and $q = 1$, we find $\|u\|_{L^2} \leq C\|G\|_{L^1}\|v\|_{L^2}$. If we take $0 < \delta < \pi/2$, then for $\lambda \in S_{\pi/2+\delta}$ we'll have $\text{Re}\sqrt{\lambda} > 0$. Hence, if $\lambda = re^{\text{i}\theta}$,

$$\|G\|_{L^1} \leq \frac{1}{|2\sqrt{\lambda}|}\int_{-\infty}^{\infty} e^{-\text{Re}\sqrt{\lambda}|z|}\text{d}z = \frac{1}{\sqrt{\lambda}\text{Re}\sqrt{\lambda}} \leq \frac{1}{|\lambda|}\frac{1}{\cos(\theta/2)} \leq \frac{M}{|\lambda|}.$$

4.2 Perturbation of the Laplacian

Definition 4.1. *Let $A : D(A) \subset X \to X$ and $B : D(B) \subset X \to X$ be linear operators. B is said to be relatively bounded with respect to A, or A-bounded, if $D(A) \subset D(B)$ and there exists positive constants α, β such that $\|Bu\| \leq \alpha\|Au\| + \beta\|u\|$ for all $u \in D(A)$. The A-bound of B is the infimum of all α such that there exists a β so that a bound of this form holds.*

Theorem 3. *[EN00, §III.2, Thm 2.10] Let A with domain $D(A)$ generate an analytic semigroup on a Banach space X. There exists a constant $\alpha_0 > 0$ such that $A + B$ with domain $D(A)$ generates an analytic semigroup for every A-bounded operator B with A-bound less than α_0.*

In order to apply this to the operator L in (4.1), first note that, since $c(x)$ is bounded $\|c(\cdot)u\|_{L^2} \leq C\|u\|_{L^2}$ for some C. So the issue is to control the first derivative term by the operator ∂_x^2. Recall that the Fourier transform is an isometry in L^2, and it sends $\partial_x^k u(x)$ to $(\mathrm{i}\xi)^k \hat{u}(\xi)$. Therefore,

$$\begin{aligned} \|u_x\|_{L^2} &= \left(\int |u_x(x)|^2 \mathrm{d}x\right)^{1/2} = \left(\int |\xi^2||\hat{u}(\xi)|^2 \mathrm{d}\xi\right)^{1/2} \leq \left(\int |\hat{u}(\xi)|^2 \mathrm{d}\xi\right)^{1/4} \left(\int |\xi|^4 |\hat{u}(\xi)|^2 \mathrm{d}\xi\right)^{1/4} \\ &\leq \frac{1}{2\epsilon}\|u\|_{L^2} + \frac{\epsilon}{2}\|\partial_x^2 u\|_{L^2}, \end{aligned}$$

Since $b(x)$ is smooth and bounded, we can therefore make the ∂_x^2-bound as small as we like. Thus, L generates an analytic semigroup, and the spectral mapping theorem holds.

4.3 Exercise: pulses of scalar reaction-diffusion equations are always linearly unstable

Consider a scalar reaction diffusion equation

$$u_t = u_{xx} + f(u),$$

and suppose that there exist a pulse-type solution, ie a function $u_*(x)$ such that $u_*'(x) \geq 0$ for all $x < x_0$, $u_*'(x) \leq 0$ for all $x > x_0$, and $u_*(x) \to 0$ exponentially fast as $|x| \to 0$. (As an example, consider §3.1.1, above.) Show that this solution is linearly unstable using the following steps.

- Show that the linearization is $v_t = v_{xx} + Df(u_*(x))v$.
- Show that $u_*'(x)$ is an eigenfunction with eigenvalue zero.
- Apply Sturm-Liouville theory to the equation $\lambda u = Lu$.
- Explain, using semigroup theory, why this implies that the pulse is linearly unstable.

Remark: this example was shown to me by Tasso Kaper.

5 Final remarks

Hopefully the above discussion has convinced you that semigroup theory is a useful tool in stability analysis. Although one must be careful to determine if the spectral mapping theorem holds in any particular setting, if it does apply then stability can be determined simply by computing the spectrum of the linear operator. This leads to one very nontrivial question: how do you compute the spectrum of a linear operator?

We saw several examples above (the Laplacian, the pulse in 3.1.1 and the traveling wave of Burgers equation in 3.1.2) where one could explicitly determine the spectrum. Although the techniques used there can be extended in some ways, for example to constant coefficient operators or other simple second-order scalar equations, in general it is very difficult to determine the spectrum of a linear operator. One tool that people often use to do this is the Evans function. If you're interested in reading about this further, see, for example, the review paper [San02]. However, it is possible that computing the spectrum is more difficult than proving linear stability by another method, which does not necessarily involve semigroups.

For example, one could use energy estimates. To illustrate this, consider again the equation $u_t = u_{xx}$ on the real line. One can compute

$$\frac{d}{dt}\frac{1}{2}\int u^2(x)\mathrm{d}x = -\int u_x^2(x)\mathrm{d}x,$$

to see that the L^2 norm of solutions is actually bounded. The above semigroup theory only tells us that the evolution in L^2 is not growing exponentially. In fact, energy estimates can be used further to show that solutions in L^2 are actually decaying algebraically and to determine the exact algebraic rate at which they decay. (One can also prove this, of course, using the explicit formula for solutions via the heat kernel.) Therefore, in some situations one can obtain much more information about asymptotic behavior from energy estimates than from semigroup theory.

References

[Bre11] Haim Brezis. *Functional analysis, Sobolev spaces and partial differential equations.* Universitext. Springer, New York, 2011.

[BSS87] J. L. Bona, P. E. Souganidis, and W. A. Strauss. Stability and instability of solitary waves of Korteweg-de Vries type. *Proc. Roy. Soc. London Ser. A*, 411(1841):395–412, 1987.

[Chi06] Carmen Chicone. *Ordinary differential equations with applications*, volume 34 of *Texts in Applied Mathematics.* Springer, New York, second edition, 2006.

[CL99] Carmen Chicone and Yuri Latushkin. *Evolution semigroups in dynamical systems and differential equations*, volume 70 of *Mathematical Surveys and Monographs.* American Mathematical Society, Providence, RI, 1999.

[DDvGV03] Gianne Derks, Arjen Doelman, Stephan A. van Gils, and Timco Visser. Travelling waves in a singularly perturbed sine-Gordon equation. *Phys. D*, 180(1-2):40–70, 2003.

[EN00] Klaus-Jochen Engel and Rainer Nagel. *One-parameter semigroups for linear evolution equations*, volume 194 of *Graduate Texts in Mathematics.* Springer-Verlag, New York, 2000. With contributions by S. Brendle, M. Campiti, T. Hahn, G. Metafune, G. Nickel, D. Pallara, C. Perazzoli, A. Rhandi, S. Romanelli and R. Schnaubelt.

[JM12] C. K. R. T. Jones and R Marangell. The spectrum of traveling wave solutions to the sine-Gordon equation. *Preprint*, 2012.

[Mik98] Milan Miklavčič. *Applied functional analysis and partial differential equations.* World Scientific Publishing Co. Inc., River Edge, NJ, 1998.

[MS61] L. D. Mešalkin and Ja. G. Sinaĭ. Investigation of the stability of a stationary solution of a system of equations for the plane movement of an incompressible viscous liquid. *J. Appl. Math. Mech.*, 25:1700–1705, 1961.

[Paz83] A. Pazy. *Semigroups of linear operators and applications to partial differential equations*, volume 44 of *Applied Mathematical Sciences.* Springer-Verlag, New York, 1983.

[San] B Sandstede. The evans function: An example.

[San02] Björn Sandstede. Stability of travelling waves. In *Handbook of dynamical systems, Vol. 2*, pages 983–1055. North-Holland, Amsterdam, 2002.

[Zum11] Kevin Zumbrun. Instantaneous shock location and one-dimensional nonlinear stability of viscous shock waves. *Quart. Appl. Math.*, 69(1):177–202, 2011.