Stability, the Maslov Index, and Spatial Dynamics

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Stability for PDEs

General framework:

$$w_t = \mathcal{F}(w), \qquad \varphi = \text{stationary solution of interest}, \qquad \mathcal{F}(\varphi) = 0$$

Analyze behavior of perturbations: $w(x,t) = \varphi(x) + u(x,t)$ with $u(x,0)$ small

$$u_t = \mathcal{L}u + \mathcal{N}(u), \qquad \mathcal{L} = D\mathcal{F}(\varphi)$$

Stability of φ: does the perturbation $u(t) \to 0$ as $t \to \infty$ (or stay small $\forall t$)?

Types of stability:

- Spectral stability: $\lambda \in \sigma(\mathcal{L}) \quad \Rightarrow \quad \mathrm{Re}(\lambda) < 0$?
- Linear stability: $u_t = \mathcal{L}u \quad \Rightarrow \quad \|u(t)\| \to 0$ as $t \to \infty$?
- Nonlinear stability: $u_t = \mathcal{L}u + \mathcal{N}(u) \quad \Rightarrow \quad \|u(t)\| \to 0$ as $t \to \infty$?

Focus on spectral stability for this talk.

Example to keep in mind

Reaction diffusion equation with gradient nonlinearity:

$$w_t = \Delta w + \nabla G(w), \qquad x \in \Omega \subset \mathbb{R}^d, \qquad w \in \mathbb{R}^n, \qquad G : \mathbb{R}^n \to \mathbb{R}$$

Solution of interest: localized stationary solution

$$0 = \Delta \varphi + \nabla G(\varphi), \qquad \lim_{|x| \to \partial\Omega} \varphi(x) = 0$$

Perturbation Ansatz: $w(x,t) = \varphi(x) + u(x,t)$

$$u_t = \mathcal{L}u + \mathcal{N}(u)$$
$$\mathcal{L}u = \Delta u + \nabla^2 G(\varphi(x))u$$
$$\mathcal{N}(u) = \nabla G(\varphi + u) - \nabla G(\varphi) - \nabla^2 G(\varphi(x))u = \mathcal{O}(u^2).$$

Spectral stability: $\sigma(\mathcal{L}) = \sigma_{ess}(\mathcal{L}) \cup \sigma_{pt}(\mathcal{L})$

- Essential spectrum relatively easy to compute; assume it is stable.
- Are there unstable eigenvalues?

Case 1: one spatial dimension, scalar equation

Sturm-Liouville eigenvalue problem:

$$\lambda u = u_{xx} + g''(\varphi(x))u = \mathcal{L}u, \qquad x \in (a, b)$$
$$u(a) = u(b) = 0$$

Prüfer coordinates: define (r, θ) via

$$u(x;\lambda) = r(x;\lambda)\sin\theta(x;\lambda), \qquad u'(x;\lambda) = r(x;\lambda)\cos\theta(x;\lambda)$$

To obtain

$$\begin{aligned} r' &= r(1 + \lambda - g''(\varphi(x)))\cos\theta\sin\theta \\ \theta' &= \cos^2\theta + (g''(\varphi(x)) - \lambda)\sin^2\theta \end{aligned}$$

Observe:

- $\{r = 0\}$ is invariant, so for a nontrivial solution,

$$u(x;\lambda) = 0 \qquad \text{if and only if} \qquad \theta = j\pi, \quad j \in \mathbb{Z}$$

- For $\lambda \ll -1$, $\theta' > 0$, so solutions will be forced to oscillate

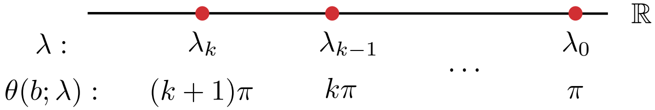
Let $\theta(a;\lambda) = 0$ be the "initial condition", evolve in x; is $\theta(b;\lambda) \in \{j\pi\}$? If so, this corresponds to an eigenfunction with eigenvalue λ.

Case 1: one spatial dimension, scalar equation

Looking for eigenfunctions and eigenvalues via

$$\theta' = \cos^2\theta + (g''(\varphi(x)) - \lambda)\sin^2\theta, \qquad x \in (a, b)$$

- Initial condition: $\theta(a;\lambda) = 0$; flow forward and see if $\theta(b;\lambda) \in \{j\pi\}$
- For some $\lambda \ll -1$ there must be an eigenvalue. Fix such a λ_k: $\theta(b;\lambda_k) = (k+1)\pi$.
- Increase λ until you again land in $\{j\pi\}$, which is the eigenvalue λ_{k-1}.



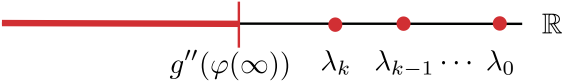
- Process stops at largest λ_0; θ no longer can complete one half-rotation

Case 1: one spatial dimension, scalar equation

Using these ideas one can show:

$$\lambda u = u_{xx} + g''(\varphi(x))u = \mathcal{L}u, \qquad x \in \mathbb{R}, \qquad u \in L^2(\mathbb{R})$$

- There exists a decreasing sequence of simple eigenvalues



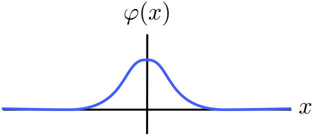
- Corresponding eigenfunctions $u_k(x)$ have k simple zeros

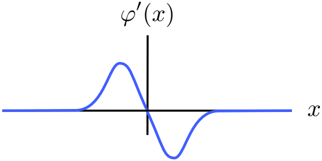
Quickly conclude the pulse is unstable: $u_t = u_{xx} + g'(u)$

- Observe that

$$\partial_x[0 = \varphi_{xx} + g'(\varphi)] \qquad 0 = (\varphi')_{xx} + g''(\varphi)\varphi' = \mathcal{L}\varphi'$$

- Qualitatively, φ and φ' look like





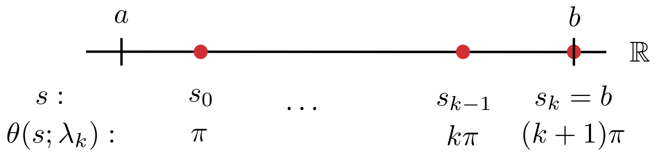
- Therefore, $\varphi' = u_1$ and $\lambda_1 = 0$, and so $\lambda_0 > 0$ and pulse is unstable

Case 1: one spatial dimension, scalar equation

Related concept of conjugate points:

$$\theta' = \cos^2\theta + (g''(\varphi(x)) - \lambda)\sin^2\theta, \qquad x \in (a, s)$$

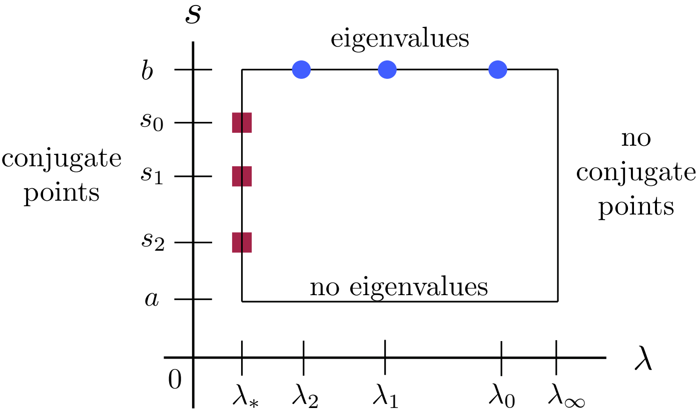
- Initial condition: $\theta(a;\lambda) = 0$; flow forward and see if $\theta(s;\lambda) \in \{j\pi\}$
- Fix $\lambda = \lambda_k$ to be an eigenvalue, so if $s = b$ we know $\theta(b;\lambda) = (k+1)\pi$
- Decrease s until you again land in $\{j\pi\}$, which is the conjugate point s_{k-1}.



- Process stops at largest s_0; θ no longer can complete one half-rotation

Case 1: one spatial dimension, scalar equation

"Square": Relationship between eigenvalues and conjugate points:

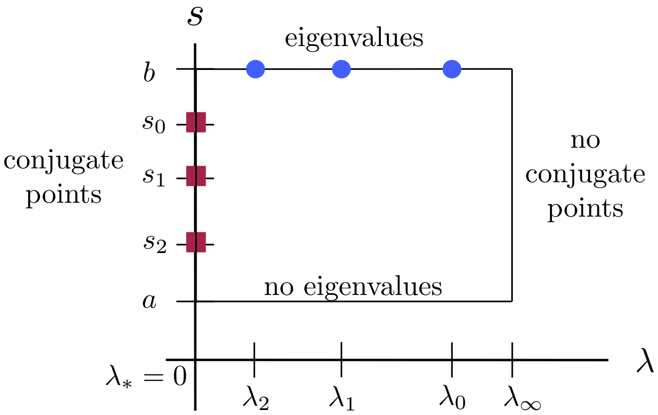


One can prove:

- No eigenvalues for $s = a$; no "time" to oscillate.
- No conjugate points for $\lambda = \lambda_\infty$ large; ODE or spectral analysis.
- Number of conjugate points for $\lambda = \lambda_*$ equals the number of eigenvalues $\lambda > \lambda_*$.

Case 1: one spatial dimension, scalar equation

To analyze stability, choose $\lambda_* = 0$:



Number of conjugate points = number of unstable eigenvalues = Morse($\mathcal{L}$)

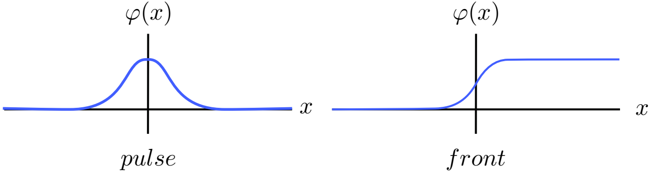
This is a simple case of what is often called the Morse Index Theorem, and it goes back to the work of Morse, Bott, etc.

Case 1: one spatial dimension, scalar equation

To summarize, when we have

$$\lambda u = u_{xx} + g''(\varphi(x))u = \mathcal{L}u, \qquad u \in \mathbb{R}, \qquad x \in \mathbb{R}$$

- Spectral stability can be determined almost immediately using qualitative properties of the underlying solution φ, with little knowledge of g.
- A pulse is necessarily unstable. Similarly a front in necessarily stable.



- Positive eigenvalues can be counted by instead counting the number of conjugate points when $\lambda = 0$.
- Conjugate points and eigenvalues can be analyzed via the winding of a phase in $\mathbb{R}^2$.
- Monotonicity in λ and s is key.

Can this be generalized to $u \in \mathbb{R}^n$ or $x \in \mathbb{R}^d$?

Case 2: one spatial dimension, system of equations

Eigenvalue equation:

$$\lambda u = u_{xx} + \nabla^2 G(\varphi(x))u = \mathcal{L}u, \qquad u(x) \in \mathbb{R}^n, \qquad x \in \mathbb{R}$$

Assume:

- $X = L^2(\mathbb{R})$, $\mathrm{dom}(\mathcal{L}) = H^2(\mathbb{R})$.
- φ is a pulse, $\lim_{|x|\to\infty} \varphi(x) = \varphi_0$.
- $\sigma(\nabla^2 G(\varphi_0)) < 0$.

Which implies

- $\mathcal{L}$ is self-adjoint, $\lambda \in \mathbb{R}$.
- $\sigma_{ess}(\mathcal{L})$ is stable.

Write as a first-order system:

$$\begin{aligned} \frac{d}{dx}\begin{pmatrix} u \\ v \end{pmatrix} &= \begin{pmatrix} 0 & I \\ (\lambda - \nabla^2 G(\varphi(x))) & 0 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} \\ &= \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}\begin{pmatrix} (\lambda - \nabla^2 G(\varphi(x))) & 0 \\ 0 & -I \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} \end{aligned}$$

[Arnol'd '85]: generalized notion of phase via the Maslov index and proved oscillation theorems

Case 2: one spatial dimension, system of equations

First-order eigenvalue problem:

$$\frac{d}{dx}\begin{pmatrix} u \\ v \end{pmatrix} = J\mathcal{B}(x;\lambda)\begin{pmatrix} u \\ v \end{pmatrix},$$

$$J = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}, \quad \mathcal{B}(x;\lambda) = \begin{pmatrix} (\lambda - \nabla^2 G(\varphi(x))) & 0 \\ 0 & -I \end{pmatrix}, \qquad \mathcal{B}(x;\lambda)^* = \mathcal{B}(x;\lambda)$$

Assumption $\sigma(\nabla^2 G(\varphi_0)) < 0$ implies $J\mathcal{B}_\infty(\lambda)$ is hyperbolic:

$$\begin{pmatrix} 0 & I \\ (\lambda - \nabla^2 G(\varphi_0)) & 0 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = \nu\begin{pmatrix} u \\ v \end{pmatrix} \quad \Rightarrow \quad (\lambda - \nabla^2 G(\varphi_0))u = \nu^2 u.$$

If $\mu_j \in \sigma(\nabla^2 G(\varphi_0))$, $\mu_j < 0$, $j = 1, \dots n$, then for $\lambda > 0$

$$\nu_j^\pm = \pm\sqrt{\lambda - \mu_j}, \qquad \nu_j^+ > 0 > \nu_j^-, \qquad j = 1, \dots, n.$$

Dimension of asymptotic stable/unstable subspaces is n: $\dim(\mathbb{E}_\infty^{s,u}(\lambda)) = n$

Case 2: one spatial dimension, system of equations

First-order eigenvalue problem:

$$\frac{d}{dx}\begin{pmatrix} u \\ v \end{pmatrix} = J\mathcal{B}(x;\lambda)\begin{pmatrix} u \\ v \end{pmatrix},$$

$$J = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}, \quad \mathcal{B}(x;\lambda) = \begin{pmatrix} (\lambda - \nabla^2 G(\varphi(x))) & 0 \\ 0 & -I \end{pmatrix}, \qquad \mathcal{B}(x;\lambda)^* = \mathcal{B}(x;\lambda)$$

For an eigenfunction $(u,v)(x;\lambda)$ we need $\lim_{x\to-\infty}(u,v)(x;\lambda) = (0,0)$.

$$\mathbb{E}^u_-(x;\lambda) = \left\{ \begin{pmatrix} u \\ v \end{pmatrix}(x;\lambda) \text{ solution : } \begin{pmatrix} u \\ v \end{pmatrix}(x;\lambda) \to \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ as } x \to -\infty \right\}.$$

Space of solutions asymptotic to the unstable subspace of $\mathbb{E}^u_\infty(\lambda)$.

This is a Lagrangian subspace with respect to the symplectic form

$$\omega(U,V) = \langle U, JV\rangle_{\mathbb{R}^{2n}}.$$

Lagrangian-Grassmanian:

$$\Lambda(n) = \{\ell \subset \mathbb{R}^{2n} : \dim(\ell) = n, \quad \omega(U,V) = 0 \quad \forall U,V \in \ell\}.$$

Case 2: one spatial dimension, system of equations

$$\frac{d}{dx}\begin{pmatrix} u \\ v \end{pmatrix} = J\mathcal{B}(x;\lambda)\begin{pmatrix} u \\ v \end{pmatrix}, \quad \mathcal{B}(x;\lambda)^* = \mathcal{B}(x;\lambda), \quad J^* = -J = J^{-1}$$

If $U, V \in \mathbb{E}_-^u(x;\lambda)$, then

$$\begin{aligned} \frac{d}{dx}\omega(U(x), V(x)) &= \langle U'(x), JV(x)\rangle + \langle U(x), JV'(x)\rangle \\ &= \langle J\mathcal{B}U(x), JV(x)\rangle + \langle U(x), J^2\mathcal{B}V(x)\rangle \\ &= \langle \mathcal{B}U(x), V(x)\rangle - \langle \mathcal{B}U(x), V(x)\rangle = 0. \end{aligned}$$

Moreover,

$$\lim_{x\to-\infty} U(x), V(x) = 0 \quad \Rightarrow \quad \lim_{x\to-\infty} \omega(U(x), V(x)) = 0$$

and so

$$\omega(U(x), V(x)) = 0 \qquad \forall x \in \mathbb{R}.$$

Case 2: one spatial dimension, system of equations

[B., Cox, Jones, Latushkin, McQuighan, Suhktayev '18]:

- Proved "square" relating eigenvalues to conjugate points.
- Proved a pulse solution is necessarily unstable.

Key ideas in paper:

$$\lambda u = u_{xx} + \nabla^2 G(\varphi(x))u = \mathcal{L}u, \qquad u \in \mathbb{R}^n, \qquad \mathrm{dom}(\mathcal{L}) = H^2(\mathbb{R}) \subset L^2(\mathbb{R}).$$

Compactify domain:

$$\sigma(s) = \tanh(s), \qquad s(\sigma) = \frac{1}{2}\ln\left(\frac{1+\sigma}{1-\sigma}\right), \qquad s \in [-\infty, \infty], \qquad \sigma \in [-1, 1]$$

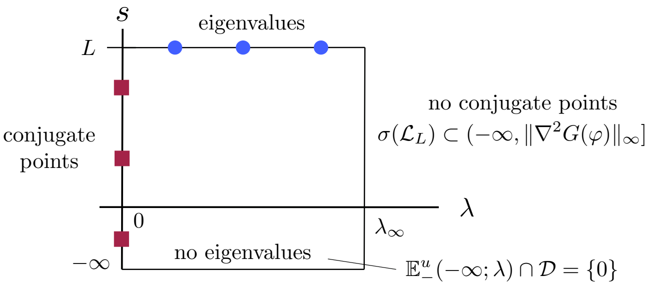
Will not comment on this further.

Case 2: one spatial dimension, system of equations

First prove "square" on half-line with Dirichlet BCs:

$$\lambda u = u_{xx} + \nabla^2 G(\varphi(x))u = \mathcal{L}_L u, \qquad x \in (-\infty, L)$$

$$\mathrm{dom}(\mathcal{L}_L) = \{u \in H^2(-\infty, L) : u(L) = 0\}.$$



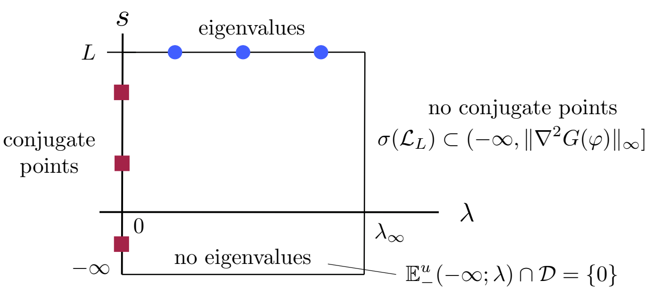
$\mathbb{E}^u_-(s;\lambda)$ is a path of Lagrangian planes, homotopic to the trivial loop, for

$$(s,\lambda) \in [-\infty, L] \times [0, \lambda_\infty]$$

Reference Lagrangian plane - Dirichlet plane:

$$\mathcal{D} = \begin{pmatrix} 0 \\ I \end{pmatrix} = \left\{ \begin{pmatrix} u \\ v \end{pmatrix} \in \mathbb{R}^{2n} : u = 0 \right\}.$$

Case 2: one spatial dimension, system of equations



Maslov index: counts crossings of the path $\mathbb{E}^u_-(s;\lambda)$ with the reference plane $\mathcal{D}$.

$$\text{Homotopy argument} \quad \Rightarrow \quad \text{Mas}(\mathbb{E}^u_-(s;\lambda)) = 0.$$

No crossings on bottom or right side of square; eigenvalues contribute negatively, conjugate points contribute positively:

$$\text{Morse}(\mathcal{L}_L) = \text{number of conjugate points on } (-\infty, L)$$

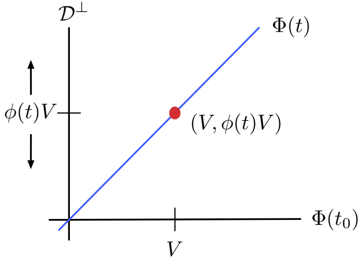
Case 2: one spatial dimension, system of equations

$\Phi : [a, b] \to \Lambda(n)$ path of Lagrangian planes, $\mathcal{D}$ reference plane. A crossing is a $t_0 \in [a, b]$ such that

$$\Phi(t_0) \cap \mathcal{D} \neq \{0\}.$$

Generically $\Phi(t)$ is transversal to $\mathcal{D}^\perp$ for all $t \in [t_0 - \epsilon, t_0 + \epsilon]$, and $\exists$ $\phi(t) : \Phi(t_0) \to \mathcal{D}^\perp$ so that

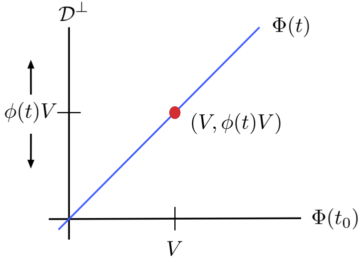
$$\Phi(t) = \text{graph } \phi(t) = \{V + \phi(t)V : V \in \Phi(t_0)\}$$



Crossing form [Robbin, Salamon '93]:

$$Q(U, V) = \frac{d}{dt}\omega(U, \phi(t)V)|_{t=t_0}, \qquad U, V \in \Phi(t_0) \cap \mathcal{D}.$$

Case 2: one spatial dimension, system of equations



Crossing form [Robbin, Salamon '93]:

$$Q(U,V) = \frac{d}{dt}\omega(U, \phi(t)V)|_{t=t_0}, \qquad U, V \in \Phi(t_0) \cap \mathcal{D}.$$

- $Q \in \mathbb{R}^{k \times k}$ symmetric, where $k = \dim(\Phi(t_0) \cap \mathcal{D})$.
- t_0 is regular if $\det Q \neq 0$; generic crossings are regular and isolated.
- Signature of Q:

$$\operatorname{sign} Q = n_+(Q) - n_-(Q),$$

$n_\pm(Q)$ = number of positive/negative eigenvalues

Case 2: one spatial dimension, system of equations

Maslov index for single crossing: if $t_0 \in [a_0, b_0]$ is the only crossing of Φ with $\mathcal{D}$,

$$\mathrm{Mas}(\Phi|_{[a_0,b_0]}, \mathcal{D}) = \begin{cases} -n_-(Q) & \text{if } t_0 = a_0 \\ \operatorname{sign} Q = n_+(Q) - n_-(Q) & \text{if } t_0 \in (a_0, b_0) \\ n_+(Q) & \text{if } t_0 = b_0 \end{cases}$$

- Endpoint convention is somewhat arbitrary; affects intermediate results but not our end result.
- Define Maslov index of a regular smooth path by defining it on segments around each crossing and summing.

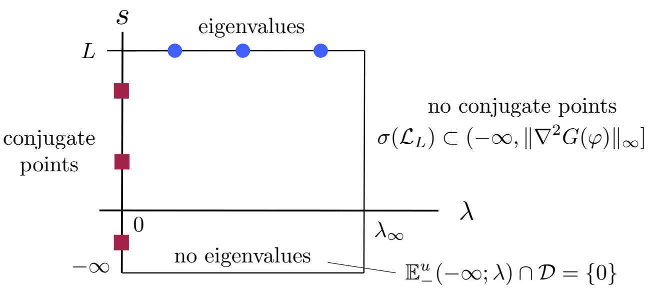
If all crossings of a path $\Phi : [a, b] \to \Lambda(n)$ with $\mathcal{D}$ are positive, ie $Q > 0$, then

$$\mathrm{Mas}(\Phi|_{[a,b]}, \mathcal{D}) = \sum_{a < t \leq b} \dim(\Phi(t) \cap \mathcal{D})$$

Similarly, if all crossings are negative, then

$$\mathrm{Mas}(\Phi|_{[a,b]}, \mathcal{D}) = \sum_{a \leq t < b} \dim(\Phi(t) \cap \mathcal{D})$$

Case 2: one spatial dimension, system of equations



Key aspect of proof is showing monotonicity, ie crossings at conjugate points are positive, and crossings at eigenvalues are negative.

Path of Lagrangian planes: $\mathbb{E}^u_-(s;\lambda)$. Parameter is s or λ depending on side.

Negative crossings in λ: need to show for $s = L$ fixed

$$Q(U,V) = \frac{d}{d\lambda}\omega(U, \phi(\lambda)V)|_{\lambda=\lambda_0} < 0, \qquad U, V \in \mathbb{E}^u(L;\lambda_0) \cap \mathcal{D}.$$

Case 2: one spatial dimension, system of equations

Suffices to check that

$$Q(V,V) = \frac{d}{d\lambda}\omega(V,\phi(\lambda)V)|_{\lambda=\lambda_0} < 0, \qquad V \in \mathbb{E}^u(L;\lambda_0) \cap \mathcal{D}.$$

Let $W(L;\lambda) \in \mathbb{E}^u(L;\lambda)$ so that

$$W(L;\lambda_0) = V, \qquad W(L;\lambda) = V + \phi(\lambda)V.$$

We have

$$\begin{aligned} Q(V,V) &= \frac{d}{d\lambda}\omega(V,\phi(\lambda)V)|_{\lambda=\lambda_0} = \frac{d}{d\lambda}\omega(V, V+\phi(\lambda)V)|_{\lambda=\lambda_0} \\ &= \frac{d}{d\lambda}\omega(W(L;\lambda_0), W(L;\lambda))|_{\lambda=\lambda_0} = \omega(W(L;\lambda_0), W_\lambda(L;\lambda_0)). \end{aligned}$$

Recall:

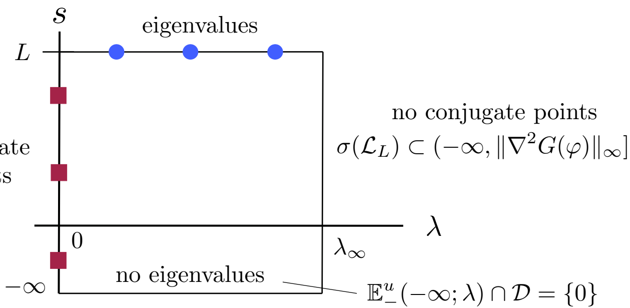
$$\frac{d}{dx}W = J\mathcal{B}(x;\lambda)W \qquad \Rightarrow \qquad \frac{d}{dx}W_\lambda = J\mathcal{B}(x;\lambda)W_\lambda + MW, \qquad M = \begin{pmatrix} 0 & 0 \\ I & 0 \end{pmatrix}.$$

Case 2: one spatial dimension, system of equations

$$Q = \omega(W(L;\lambda_0), W_\lambda(L;\lambda_0)), \qquad \frac{d}{dx} W_\lambda = J\mathcal{B}(x;\lambda) W_\lambda + MW.$$

$$\begin{aligned}
Q &= \langle -JW(L;\lambda_0), W_\lambda(L;\lambda_0))\rangle = -\int_{-\infty}^{L} \frac{d}{dx}\langle JW(x;\lambda_0), W_\lambda(x;\lambda_0))\rangle dx \\
&= -\int_{-\infty}^{L} [\langle J^2\mathcal{B}W, W_\lambda)\rangle + \langle JW, J\mathcal{B}W_\lambda + MW\rangle] dx \\
&= -\int_{-\infty}^{L} \langle JW, MW\rangle dx \\
&= -\int_{-\infty}^{L} \left\langle \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} W, \begin{pmatrix} 0 & 0 \\ I & 0 \end{pmatrix} W \right\rangle dx \\
&= -\int_{-\infty}^{L} (W_1(x;\lambda_0))^2 dx < 0.
\end{aligned}$$

Case 2: one spatial dimension, system of equations



This monotonicity implies:

$$
\begin{aligned}
0 &= \mathrm{Mas}(\mathbb{E}^u(x;\lambda)_{\mathrm{square}}, \mathcal{D}) \\
&= \mathrm{Mas}(\mathbb{E}^u(x;\lambda)_{\mathrm{left}}, \mathcal{D}) + \mathrm{Mas}(\mathbb{E}^u(x;\lambda)_{\mathrm{top}}, \mathcal{D}) + 0 + 0 \\
&= \{\text{number of conjugate points}\} - \{\text{number of eigenvalues}\}.
\end{aligned}
$$

Hence,

$$\{\text{number of conjugate points}\} = \{\text{number of eigenvalues}\} = \mathrm{Morse}(\mathcal{L}_L).$$

Case 2: one spatial dimension, system of equations

Remaining steps:

$$\lambda u = u_{xx} + \nabla^2 G(\varphi(x))u = \mathcal{L}u, \qquad u \in \mathbb{R}^n, \qquad x \in \mathbb{R}.$$

Extend result to full line $\mathbb{R}$: show for $L > L_\infty$ large,

- $\mathrm{Morse}(\mathcal{L}) = \mathrm{Morse}(\mathcal{L}_L)$
- Follows because you can approximate the point spectrum of an operator on $\mathbb{R}$ using a large subdomain.

Prove any pulse solution is necessarily unstable:

- Show there is at least one conjugate point.
- Uses reversibility arguments applied to the original PDE $u_t = u_{xx} + \nabla G(u)$, ie $x \to -x$ symmetry.

Remark: very few actual applications of the Maslov index in stability analysis!

Case 3: multiple spatial dimensions, one equation

Eigenvalue problem:

$$\Delta u + V(x)u = \lambda u, \qquad x \in \Omega \subset \mathbb{R}^d, \qquad u \in \mathbb{R}^n, \qquad \lambda \in \mathbb{R}$$
$$u|_{\partial\Omega} = 0$$

Family of domains [Smale 65]:

$$\{\Omega_s : 0 \le s \le 1\}, \qquad \Omega_1 = \Omega, \qquad \Omega_0 = \{x_0\}.$$

Hilbert space

$$\mathcal{H} = H^{1/2}(\partial\Omega) \times H^{-1/2}(\partial\Omega), \qquad \omega((f_1, g_1), (f_2, g_2)) = \langle g_2, f_1 \rangle - \langle g_1, f_2 \rangle$$

Path of subspaces in the Fredholm-Lagrangian Grassmannian of $\mathcal{H}$:

$$\Phi(s) = \left\{ \left(u, \frac{\partial u}{\partial n} \right) |_{\partial\Omega_s} : u \in H^1(\Omega_s), \quad \Delta u + V(x)u = \lambda u, \quad x \in \Omega_s \right\}$$

Reference subspace:

$$\mathcal{D} = \left\{ \left(u, \frac{\partial u}{\partial n} \right) |_{\partial\Omega} = \left(0, \frac{\partial u}{\partial n} \right) |_{\partial\Omega} : u \in H^1(\Omega_s) \right\}$$

[Deng, Jones '11], [Cox, Jones, Latushkin, Suhktayev '16], ... show one can compute $\mathrm{Morse}(\mathcal{L})$ by counting conjugate points in this context; also results for more general boundary conditions.

Case 3: multiple spatial dimensions, one equation

Future work: does this suggest a "spatial dynamics" for $\mathbb{R}^d$?

$$0 = \Delta u + F(u), \qquad x \in \Omega \subset \mathbb{R}^d$$

Family of domains parameterized by family of diffeomorphisms:

$$\psi_s : \Omega \to \Omega_s, \qquad s \in [0,1], \qquad \Omega_1 = \Omega, \qquad \Omega_0 = \{x_0\}.$$

Define boundary data via

$$f(t;y) = u(\psi_t(y)), \qquad g(t;y) = \frac{\partial u}{\partial n}(\psi_t(y)), \qquad t \in [0,1], \qquad y \in \partial\Omega$$

and trace map

$$\mathrm{Tr}_t u = (f(t), g(t)).$$

Can (formally) obtain a first-order system

$$\frac{d}{dt}\begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} \mathcal{F}(f,g) \\ \mathcal{G}(f,g) \end{pmatrix}$$

Can we make sense of the above equation and put it to good use?

Summary

We've seen how the Maslov index can be used in stability analysis to obtain results of the form

$$\text{number of conjugate points} = \text{Morse}(\mathcal{L}).$$

Due to recent work, many abstract results exist for systems of equations with $x \in \mathbb{R}^d$, $d \geq 1$.

Current/future work:

- Find more examples! In some sense, these results are only useful if they can be used to actually determine stability in situations of interest. I know of two examples for $x \in \mathbb{R}$ (one mentioned today - pulse instability; other is [Chen, Hu '14]) and none for $x \in \mathbb{R}^d$.
- Further understand relationship between these results and the Evans function.
- Develop a spatial dynamics for $\mathbb{R}^d$.

Lagrangian subspace calculation

$$\Phi(s) = \left\{ \left(u, \frac{\partial u}{\partial n} \right) |_{\partial\Omega_s} : u \in H^1(\Omega_s), \quad \Delta u + V(x)u = \lambda u, \quad x \in \Omega_s \right\}$$

If $u, v \in \Phi$ then

$$\begin{aligned}
\omega(u, v) &= \left\langle \frac{\partial v}{\partial n}, u \right\rangle - \left\langle \frac{\partial u}{\partial n}, v \right\rangle \\
&= \int_{\partial\Omega} \left(\frac{\partial v}{\partial n} u - \frac{\partial u}{\partial n} v \right) dS \\
&= \int_{\Omega} \left((\nabla u \nabla v + u\Delta v) - (\nabla u \nabla v + v\Delta u) \right) dx \\
&= \int_{\Omega} \left(u(\lambda v - Vv) - v(\lambda u - Vu) \right) dx = 0.
\end{aligned}$$

Pulse - existence of conjugate point

$$0 = \varphi_{xx} + \nabla G(\varphi(x)),$$

- Generically, $\varphi(x)$ will be unique as a solution (up to translation) asymptotic to the fixed point $\varphi_0 = \lim_{x\to\pm\infty}\varphi(x)$
- Equation invariant under $x \to -x$, so $\varphi(-x)$ is also a solution. By uniqueness, we therefore have

$$\varphi(x) = \varphi(-x+\delta)$$

- This implies

$$\varphi(\delta/2 + x) = \varphi(\delta/2 - x) \qquad \forall \quad x \in \mathbb{R}.$$

- But then

$$\frac{d}{dx}\varphi(\delta/2+x)|_{x=0} = \varphi(\delta/2-x)|_{x=0} \qquad \Rightarrow \qquad \varphi_x(\delta/2) = 0.$$

- Since φ_x is a eigenfunction with eigenvalue $\lambda = 0$, we have

$$\begin{pmatrix}\varphi_x(x)\\ \varphi_{xx}(x)\end{pmatrix} \in \mathbb{E}^u_-(x;0) \qquad \Rightarrow \qquad \mathbb{E}^u_-(\delta/2,0)\cap\mathcal{D} \neq \{0\}$$

which is our conjugate point.