Using global invariant manifolds to understand metastability in Burgers equation with small viscosity

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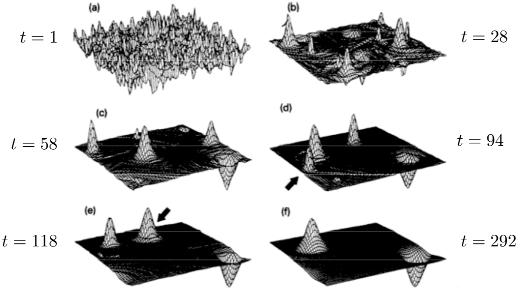
Motivation: Metastability in the Navier-Stokes Equations

$$\text{fluid velocity} = \mathbf{u}$$

$$\text{vorticity} = \omega = \nabla \times \mathbf{u}$$

$$\frac{\partial \omega}{\partial t} = \mu \triangle \omega - \mathbf{u} \cdot \nabla \omega$$

$$0 < \mu \ll 1$$



[Matthaeus et. al., Physica D, 91]

- Unbounded domains:
  - Single Oseen vortex globally stable [Gallay & Wayne 05]
  - Analysis of initial vortex motion and deformation [Gallay 09]
- Numerically observed metastability [Matthaeus et. al., 91]:
  - Solutions rapidly approach solutions to Euler equations ($\mu = 0$)
  - Large time needed for all vortices to coalesce

Motivation: Metastability in Burgers Equation

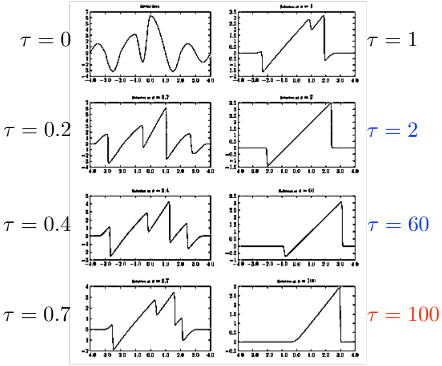
1D Burgers equation in similarity variables:

$$u_t = \mu u_{xx} - u u_x$$

$$0 < \mu \ll 1$$

Zero viscosity:
N-waves stable

Nonzero viscosity:
Diffusion waves stable



[Kim & Tzavaras, SIAM J. Math. Anal., 01]

Results from [Kim & Tzavaras 01]:

- Observed numerically
- Explained formally using asymptotic expansions

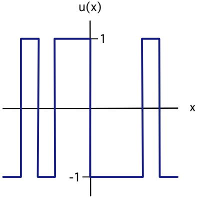
Related results

Metastability in gradient systems:

$$u_t = \epsilon^2 u_{xx} - u(u^2 - 1), x \in (0, 1)$$

Eg: [Carr & Pego 89], [Fusco & Hale 89], [Chen 04], [Otto & Reznikoff 07]

- Stable states: $u \equiv \pm 1$
- Metastable states: step functions connecting ± 1 numerous times



Mechanism appears different from Navier-Stokes and Burgers:

- Utilize gradient structure
- Pattern of transition layers can be related to

$$E[u](t) = \int_0^1 \left[\frac{\epsilon^2}{2} u_x^2 + \frac{1}{4}(u^2 - 1)^2 \right] dx$$

But 'metastable' time scales are the similar!

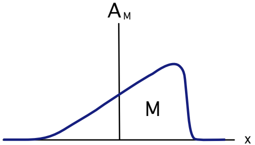
Recall known results on Burgers equation

Burgers Equation:

$$u_t = \mu u_{xx} - u u_x, \qquad x \in \mathbb{R}, \quad t > 0, \quad u \in \mathbb{R}$$
$$u(x,0) = u_0(x), \qquad 0 < \mu \ll 1$$

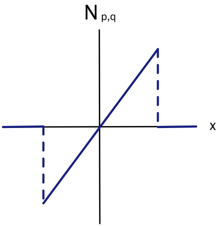
- If $\mu > 0$ and
  - u_0 smooth, localized
  - $\int u_0(x)dx = M$

then u converges in L^1 to a diffusion wave of mass M [Liu 00], [Kim & Tzavaras 01]



- If $\mu = 0$ and
  - u_0 compactly supported
  - u_0 satisfies an entropy condition

then u converges in L^1 to an N-wave [Lax 73]



Recall known results on Burgers equation

Burgers Equation:

$$\begin{aligned} u_t &= \mu u_{xx} - u u_x, & x \in \mathbb{R}, \quad t > 0, \quad u \in \mathbb{R} \\ u(x,0) &= u_0(x), & 0 < \mu \ll 1 \end{aligned}$$

• For any fixed $t > 0$,

$$\lim_{\mu \to 0} u(x,t;\mu) = u(x,t;0)$$

pointwise in x [Evans 98].

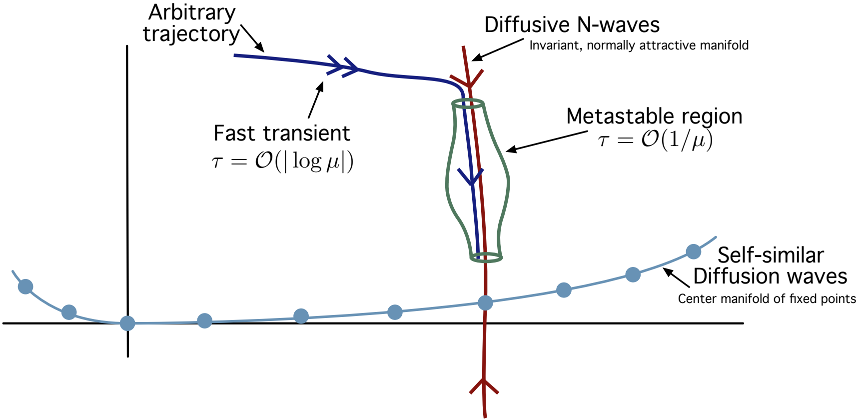
• For $0 < \mu \ll 1$, $u(x,t;\mu)$ looks like an N-wave for large times before converging to a diffusion wave. Shown numerically and using asymptotic expansions in [Kim & Tzavaras 01].

Want to understand the interplay between the limits

$$\mu \to 0, \qquad t \to \infty$$

Main results

Geometry in phase space:



Scaling or similarity variables

Burgers Equation:

$$
\begin{aligned}
u_t &= \mu u_{xx} - u u_x, \qquad x \in \mathbb{R}, \quad t > 0, \quad u \in \mathbb{R} \\
u(x,0) &= u_0(x), \qquad 0 < \mu \ll 1
\end{aligned}
$$

Scaling variables - deal with continuous spectrum:

$$
u(x,t) = \frac{1}{\sqrt{t+1}} w\left(\frac{x}{\sqrt{t+1}}, \log(t+1)\right)
$$

$$
\xi = \frac{x}{\sqrt{t+1}}, \quad \tau = \log(t+1)
$$

Scaled Burgers equation:

$$
\begin{aligned}
w_\tau &= \mu w_{\xi\xi} + \frac{1}{2}\xi w_\xi + \frac{1}{2} w - w w_\xi \\
\mathcal{L} w &= \partial_\xi^2 w + \frac{1}{2}\partial_\xi(\xi w)
\end{aligned}
$$

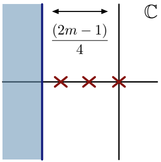
Spectrum of $\mathcal{L}$

In the space

$$L^2(m) := \left\{ w \in L^2(\mathbb{R}) : \int_{\mathbb{R}} (1+\xi^2)^m w^2(\xi) d\xi < \infty \right\}$$

the spectrum of $\mathcal{L}$ is [Gallay & Wayne 02]

$$\sigma(\mathcal{L}) = \left\{ -\frac{n}{2} : n \in \mathbb{N} \right\} \cup \left\{ \lambda \in \mathbb{C} : \mathrm{Re}\lambda \leq \frac{1-2m}{4} \right\}$$



Eigenfunctions:

$$\lambda = -\frac{n}{2}, \quad \varphi_0(\xi) = \frac{1}{\sqrt{4\pi\mu}} e^{-\frac{\xi^2}{4\mu}}, \quad \varphi_n(\xi) = (\partial_\xi^n \varphi_0)(\xi)$$

Invariant manifolds:

- 1D global center manifold: diffusion waves
- 1D global foliation: diffusive N-waves

1D global center manifold

Burgers equation:

$$w_\tau = \mu w_{\xi\xi} + \frac{1}{2}\xi w_\xi + \frac{1}{2}w - ww_\xi$$

Stationary solutions:

$$0 = \mu w_\xi + \frac{1}{2}\xi w - \frac{1}{2}w^2$$

Solve explicitly:

$$A_M(\xi) = \frac{\alpha e^{-\frac{\xi^2}{4\mu}}}{1 - \frac{\alpha}{2\mu}\int_{-\infty}^{\xi} e^{-\frac{\eta^2}{4\mu}}\, d\eta}, \qquad \alpha = \sqrt{\frac{\mu}{\pi}}\left(1 - e^{-\frac{M}{2\mu}}\right)$$

Claim: The family $\{A_M\}$ corresponds to a 1D global center manifold in $L^2(m)$ for $m > 1/2$.

Proof: Fix $m > 1/2$, so $\lambda = 0$ is isolated.

- Apply invariant manifold theorem, e.g. [Chen, Hale, & Tan 97]
- Then there exists a local 1D center manifold
- It must contain all globally bounded in time solutions.
- Therefore the manifold is global and equal to $\{A_M\}$.

1D global stable foliation

Linearize about a diffusion wave: $w(\xi,\tau) = A_M(\xi) + v(\xi,\tau)$

$$v_\tau = \underbrace{\mathcal{L}v - (A_M v)_\xi}_{\mathcal{A}_M v} - v v_\xi$$

Can show explicitly that $\mathcal{A}_M$ and $\mathcal{L}$ are conjugate:

$$\mathcal{A}_M U = U\mathcal{L}, \quad U \in L(L^2(m))$$

Therefore, their spectra coincide. Fix $m > 3/2$ to get a local, 2D center-stable manifold near each diffusion wave.

Want to show:

- Chose mass M appropriately, it is 1D
- It is a global manifold

Use Cole-Hopf.

1D global stable foliation

$$v_\tau = \mathcal{A}_M v - v v_\xi$$

Cole-Hopf:

$$V(\xi,\tau) = v(\xi,\tau) e^{-\frac{1}{2\mu}\int_{-\infty}^{\xi} v(y,\tau)dy} \quad \Rightarrow \quad V_\tau = \mathcal{A}_M V$$

Eigenfunctions:

$$\begin{aligned} \lambda &= 0, \qquad \Phi_0(\xi), \quad \Psi_0(\xi) \equiv 1 \\ \lambda &= -1/2, \quad \Phi_1(\xi) = A_M' \end{aligned}$$

- 2D center-stable subspace: span$\{\Phi_0, \Phi_1\}$
- Restrict to solutions such that: $\langle \Psi_0, V_0 \rangle = \int V_0(\xi) d\xi = 0$

$$\int V_0(\xi)d\xi = -2\mu \int \partial_\xi \left(e^{-\frac{1}{2\mu}\int_{-\infty}^{\xi} v_0(y)dy} \right) d\xi = -2\mu \left(e^{-\frac{1}{2\mu}\int v_0(y)dy} - 1 \right) = 0$$

$$\Leftrightarrow \int v_0(\xi)d\xi = 0$$

Recall $w = A_M + v$: chose $M = \int w(\xi, 0)d\xi$

$$\tilde{w}_N(\xi,\tau) = A_M(\xi) + \frac{\tilde{\alpha} e^{-\frac{\tau}{2}} A_M'(\xi)}{1 - \frac{\tilde{\alpha}}{2\mu} e^{-\frac{\tau}{2}} A_M(\xi)}$$

1D global stable foliation

Alternatively, apply Cole-Hopf directly to Burgers:

$$w_\tau = \mathcal{L}w - ww_\xi$$

$$W(\xi,\tau) = w(\xi,\tau)e^{-\frac{1}{2\mu}\int_{-\infty}^{\xi} w(y,\tau)dy} \quad \Rightarrow \quad W_\tau = \mathcal{L}W$$

2D invariant subspace: span$\{\varphi_0, \varphi_1\}$

$$W_N(\xi,\tau) = \beta_0\varphi_0(\xi) + \beta_1 e^{-\frac{\tau}{2}}\varphi_1(\xi)$$

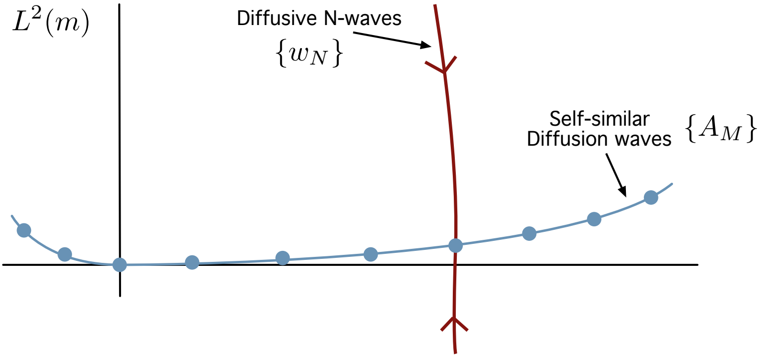
Invert Cole-Hopf:

$$w_N(\xi,\tau) = \frac{\beta_0\varphi_0(\xi) + \beta_1 e^{-\frac{\tau}{2}}\varphi_1(\xi)}{1 - \frac{\beta_0}{2\mu}\int_{-\infty}^{\xi}\varphi_0(y)dy - \frac{\beta_1}{2\mu}e^{-\frac{\tau}{2}}\varphi_0(\xi)}$$

Diffusive N-waves

Geometry of Phase Space

We now have:



Need to show:

- w_N looks like an inviscid N-wave; family is attracting
- Time scales $\mathcal{O}(|\log \mu|)$ and $\mathcal{O}(1/\mu)$

N-waves

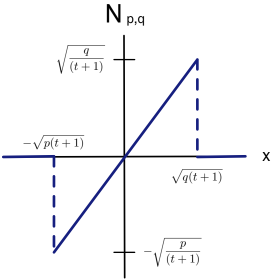
$$u_t = \mu u_{xx} - u u_x$$

For $\mu = 0$, there are invariants:

$$p = -2\ \inf_y \int_{-\infty}^{y} u(x)dx, \qquad q = 2\ \sup_y \int_{y}^{\infty} u(x)dx, \qquad 2M = q - p$$

N-wave:

$$N_{p,q}(x,t) = \begin{cases} \frac{x}{t+1} & \text{if} \quad -\sqrt{p(t+1)} < x < \sqrt{q(t+1)} \\ 0 & \text{otherwise} \end{cases}$$



Self-similar N-waves

$$w_\tau = \mu w_{\xi\xi} + \frac{1}{2}\xi w_\xi + \frac{1}{2}w - ww_\xi$$

- For $\mu = 0$, p and q still invariant.

N-wave:

$$N_{p,q}(\xi) = \begin{cases} \xi & \text{if} \quad -\sqrt{p} < \xi < \sqrt{q} \\ 0 & \text{otherwise} \end{cases}$$

- For $\mu > 0$, diffusive N-wave

$$w_N(\xi, \tau) = \frac{\beta_0 \varphi_0(\xi) + \beta_1 e^{-\frac{\tau}{2}} \varphi_1(\xi)}{1 - \frac{\beta_0}{2\mu} \int_{-\infty}^{\xi} \varphi_0(y) dy - -\frac{\beta_1}{2\mu} e^{-\frac{\tau}{2}} \varphi_0(\xi)}$$

where

$$\beta_0 = 2\mu(1 - e^{-\frac{M}{2\mu}}), \quad \beta_1 e^{-\frac{\tau}{2}} = F(p(\tau), q(\tau)), \quad 2M = q(\tau) - p(\tau)$$

N-waves near Diffusive N-waves

Lemma: Given any positive constants p, q, and δ, let $w_N(\xi, \tau)$ be the diffusive N-wave such that, at time $\tau = \tau_0$, the positive mass of $w_N(\cdot, \tau_0)$ is q and the negative mass is p. There exists a μ_0 sufficiently small so that, if $0 < \mu < \mu_0$, then

$$\|w_N(\cdot, \tau_0) - N_{p,q}(\cdot)\|_{L^2(m)} \leq \delta.$$

Proof: Calculation (lengthy) using the explicit formulas for w_N and $N_{p,q}$.

Intuitively:

$$w_N(\xi, \tau) = \frac{\xi - \frac{2\mu\beta_0}{\beta_1 e^{-\tau/2}}}{1 - \frac{2\mu}{\beta_1 e^{-\tau/2}\varphi_0(\xi)}\left[1 - \frac{\beta_0}{2\mu}\int_{-\infty}^{\xi}\varphi_0(y)dy\right]}$$

Remark: This is proven pointwise in [Kim & Tzavaras 01].

Initial Transient

Theorem: Fix $m > 3/2$. Let $w(\xi, \tau)$ be a solution to Burgers equation with mass M, and let $N_{p,q}$ be the inviscid N-wave with values p and q determined by the initial data $w_0(\xi) \in L^2(m)$. Given any $\delta > 0$, there exists a μ sufficiently small and a $T = \mathcal{O}(|\log \mu|)$ so that

$$\|w(\cdot, T) - N_{p,q}(\cdot)\|_{L^2(m)} \leq \delta.$$

Proof: Calculation (lengthy) using the formulas for w (Cole-Hopf) and $N_{p,q}$.

Remarks:

- p, q determined at $\tau = 0$, but estimate is at $\tau = T$
- This is because p, q evolve slowly.
- Timescale $\mathcal{O}(|\log \mu|)$ unexpected; similar to gradient systems.
- Timescale matches numerics of [Kim & Tzavaras 01]

Rates of Change of $p(\tau)$ and $q(\tau)$

[Kim & Tzavaras 01]: Estimate $\dot{p}$ and $\dot{q}$ within $\{w_N\}$

$$\begin{aligned} p &= -2\,\inf_y \int_{-\infty}^{y} w_N(\xi)d\xi \\ &= -2\,\inf_y \int_{-\infty}^{y} \left[-2\mu\partial_\xi \log\left(1-\frac{\beta_0}{2\mu}\int_{-\infty}^{\xi}\varphi_0(z)dz - \frac{\beta_1 e^{-\tau/2}}{2\mu}\varphi_0(\xi)\right)\right] d\xi \\ &= 4\mu\,\sup_y \log\left(1-\frac{\beta_0}{2\mu}\int_{-\infty}^{y}\varphi_0(z)dz - \frac{\beta_1 e^{-\tau/2}}{2\mu}\varphi_0(y)\right) \end{aligned}$$

Compute $y = y^*$ for which this is attained. If $M > 0$:

$$e^{\frac{p}{4\mu}} - 1 \le -\frac{\beta_1 e^{-\frac{\tau}{2}}}{2\mu}\varphi_0(y^*) \le e^{\frac{p}{4\mu}} - e^{-\frac{M}{2\mu}}$$

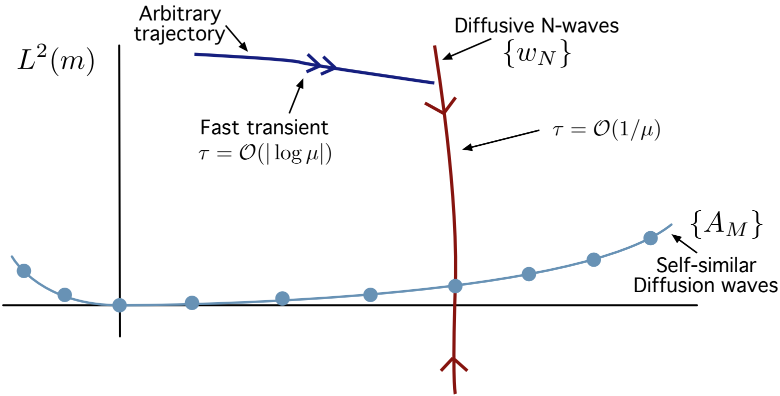
Use this to show

$$p(\tau) = \mathcal{O}(\mu\tau)$$

Timescale for $w_N \to A_M$ is same as for $p \to 0$.

Geometry of Phase Space

We now have:



Need to show:

- Existence of the metastable region

Local Attractivity

Theorem: Fix $m > 5/2$. There exists a c_0 sufficiently small such that, for any solution $w(\cdot, \tau)$ of Burgers equation with initial data

$$w(\xi, 0) = w_N(\xi, 0) + \phi_0(\xi), \quad \|\phi_0\|_{L^2(m)} \leq c_0$$

there exists a constant C_ϕ such that

$$w(\xi, \tau) = w_N(\xi, \tau) + \phi(\xi, \tau), \quad \|\phi(\cdot, \tau)\|_{L^2(m)} \leq C_\phi e^{-\tau}.$$

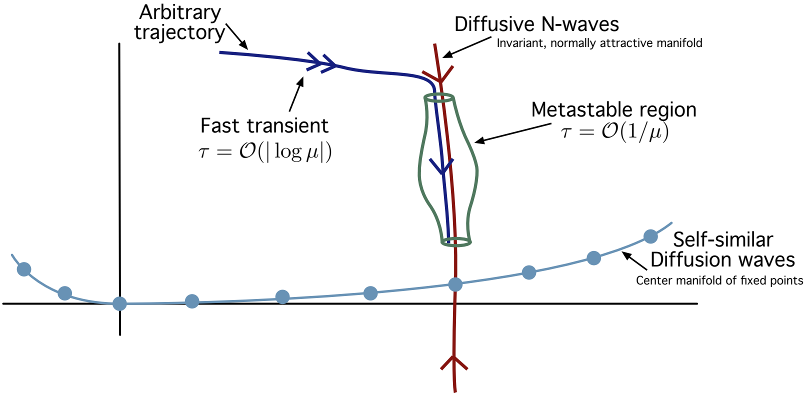
Proof: Calculation (short) using Cole-Hopf and the spectral properties of $\mathcal{L}$.

Remarks:

- Rates of change of p, q determine decay rate to A_M
- The constant C_ϕ can be large with respect to μ
- Rate $e^{-\tau}$ seems optimal based on numerics in [Kim & Ni 02]

Summary and discussion

We have shown:



Remarks:

- Metastability is not caused by dependence of $\sigma(\mathcal{L})$ on μ
- Would be nice to not use Cole-Hopf

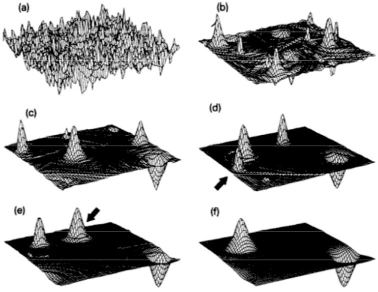
Towards understanding metastability in Navier-Stokes

fluid velocity $= \mathbf{u}$

vorticity $= \omega = \mathrm{curl}\ \mathbf{u}$

$$\frac{\partial \omega}{\partial t} = \mu \triangle \omega - \mathbf{u} \cdot \nabla \omega$$

$$0 < \mu \ll 1$$



[Matthaeus et. al., Physica D, 91]

- Unbounded domains:
  - Single Oseen vortex globally stable [Gallay & Wayne 05]
  - Analysis of initial vortex motion and deformation [Gallay 09]
- Numerically observed metastability [Matthaeus et. al., 91]:
  - Solutions rapidly approach solutions to Euler equations ($\mu = 0$)
  - Large time needed for all vortices to coalesce

Towards understanding metastability in Navier-Stokes

Recall results of [Gallay & Wayne 02, 05]:

$$\frac{\partial \omega}{\partial t} = \mu \triangle \omega - u \cdot \nabla \omega, \qquad u(x) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{(x-y)^\perp}{|x-y|^2} \omega(y) dy$$

Scaling variables:

$$\omega(x, \tau) = \frac{1}{(t+1)} w \left(\frac{x}{\sqrt{t+1}}, \log(t+1) \right)$$

$$w_\tau = \underbrace{\mu \triangle w + \tfrac{1}{2}(\xi \cdot \nabla) w + w}_{\mathcal{L} w} - (v \cdot \nabla) w$$

1D global center manifold:

$$\mathcal{L} \varphi_0 = 0, \qquad \mathcal{L} \varphi_0 - (v^{\varphi_0} \cdot \nabla) \varphi_0 = 0$$

Oseen vortices $\{\alpha \varphi_0\}$ are globally stable.

- Is there a global foliation?
- What causes the separation in time scales?

Separation in time scales

Carr/Pego example:

$$u_t = \epsilon^2 u_{xx} - u(u^2 - 1) \quad \to \quad u_t = -u(u^2 - 1).$$

- Limit is ODE; exponential growth/decay towards fixed points $u = \pm 1$.
- Time for gradients of size $1/\epsilon$ is

$$e^t \approx 1/\epsilon \quad \to \quad t = -\log \epsilon.$$

Burgers:

$$u_t = \mu u_{xx} - u u_x \quad \to \quad u_t = -u u_x.$$

- Limit is PDE; dynamics determined by motion along characteristics.
- Similarity variables induce exponential rates along them - same timescale.

Navier-Stokes:

$$\omega_t = \mu \triangle \omega - \mathbf{u} \cdot \nabla \omega \quad \to \quad \omega_t = -\left(\frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{(x-y)^\perp}{|x-y|^2} \omega(y) dy \right) \cdot \nabla \omega$$

- Limit is PDE; nonlinearity nonlocal
- Similarity variables again induce exponential behavior - how to analyze?