Nonlinear stability of coherent structures via pointwise estimates

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joint work with
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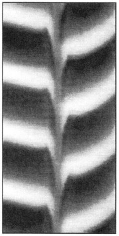
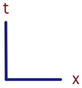
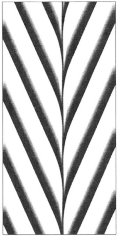
An example of a coherent structure: a source

Time-periodic patterns in reaction-diffusion systems:







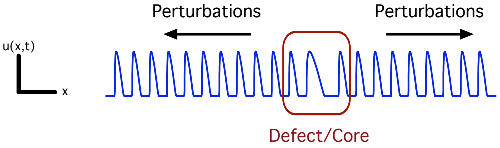
Experiment: chemical reaction
chlorite-iodite-malonic-acid (CIMA)

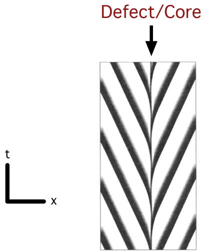
Numerical simulation:
reaction-diffusion equation
$u_t = Du_{xx} + f(u)$

[Perraud et. al., Phys. Rev. Lett. 1993]

Why study sources?

Diagram of a source:





Importance in applications:

- Widely observed in experiments and numerics
- Defect created spontaneously; not caused by inhomogeneity
- Organize dynamics in rest of spatial domain

Mathematical interest:

- Linear stability: embedded zero eigenvalue; time-periodic linear operator
- Nonlinear stability: weighted spaces don't work, estimates are delicate

Stability

Reaction diffusion equation:

$$u_t = Du_{xx} + f(u)$$

Stationary solution of interest: $u(x,t) = u^*(x)$

Ansatz: $u(x,t) = u^*(x) + v(x,t)$

$$\begin{aligned} v_t &= D(u^*_{xx} + v_{xx}) + f(u^* + v) \\ &= \underbrace{Dv_{xx} + f'(u^*(x))v}_{\mathcal{L}v} + \underbrace{[f(u^* + v) - f(u^*) - f'(u^*(x))v]}_{N(v)} \end{aligned}$$

Stability:

- Strong: $v(t) \to 0$ as $t \to \infty$
- Weak: $v(t)$ stays small for all $t \geq 0$.

Stability at the linear level

Ignore the nonlinear term:

$$v_t = \mathcal{L} v$$

Two types of stability:

- Spectral stability: $\sup \mathrm{Re}\sigma(\mathcal{L}) < 0$ (weak: $\sup \mathrm{Re}\sigma(\mathcal{L}) \leq 0$)
- Linear stability: $v(t) \to 0$ as $t \to \infty$ (weak: $v(t)$ stays small $\forall t \geq 0$)

In finite dimensions ($\mathcal{L}$ a matrix), spectral and linear stability are equivalent. In infinite dimensions ($\mathcal{L}$ is a differential operator), they may not be!

Example: in $X = H^1(1, \infty)$,

$$v_t = x \partial_x v = \mathcal{L} v$$

- $\sup \mathrm{Re}\sigma(\mathcal{L}) \leq -1/2$
- $\exists$ solution $v(t) \sim e^{t/2}$

For reaction diffusion equations, typically spectral and linear stability are equivalent.

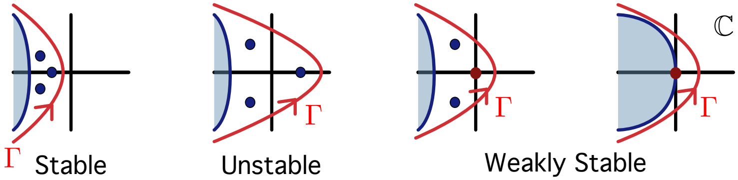
Stability at the linear level

Determine behavior of

$$v_t = \mathcal{L} v$$

Take the Laplace transform $\hat{v}(\lambda) = \int_0^\infty e^{-\lambda t} v(t) dt$

$$\lambda \hat{v} - v_0 = \mathcal{L} \hat{v} \qquad \Rightarrow \qquad v(t) = \frac{1}{2\pi i} \int_\Gamma e^{\lambda t} (\lambda - \mathcal{L})^{-1} v_0 d\lambda =: e^{\mathcal{L} t} v_0$$



Linear estimates:

- Stable: $\|e^{\mathcal{L}t}\| \leq C e^{-\delta t}$
- Weakly stable: $\|e^{\mathcal{L}t}\| \leq C$

Nonlinear stability

$$v_t = \mathcal{L}v + N(v)$$

Transfer linear decay to the nonlinear equation via

$$v(t) = e^{\mathcal{L}t}v_0 + \int_0^t e^{\mathcal{L}(t-s)}N(v(s))ds$$

If $\|e^{\mathcal{L}t}\| \leq Ce^{-\delta t}$ and $|N(v)| \leq |v|^2$, we can define

$$\|v\| = \sup_{0\leq t\leq T} e^{\delta t}|v(t)|$$

Then we find

$$\|v\| \leq C|v_0| + \int_0^T Ce^{-\delta s}\|v\|^2 ds.$$

If we now take $T = \sup\{t : e^{\delta t}|v(t)| \leq M\}$, then in fact

$$\|v\| \leq \frac{C|v_0|}{1 - \frac{C}{\delta}(1 - e^{-\delta T})\|v\|} \leq \frac{C|v_0|}{1 - \frac{C}{\delta}M} < M$$

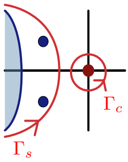
if we take M and $|v_0|$ sufficiently small. So $|v(t)| \leq Me^{-\delta t}$ for all $t \geq 0$.

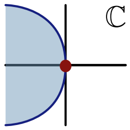
What about weak linear stability?

Expect zero eigenvalue:

$$0 = u^*_{xx} + f(u^*(x)) \qquad \Rightarrow \qquad 0 = \partial_x^2(u^*_x) + f'(u^*(x))u^*_x = \mathcal{L}u^*_x$$

Weakly stable spectrum:





?????

Spectral gap: separate into stable (decaying) and center (bounded) part:

$$e^{\mathcal{L}t} = \frac{1}{2\pi i}\int_{\Gamma_s} e^{\lambda t}(\lambda - \mathcal{L})^{-1} d\lambda + \frac{1}{2\pi i}\int_{\Gamma_c} e^{\lambda t}(\lambda - \mathcal{L})^{-1} d\lambda$$

And show (nonlinearly) the solution converges to a spatial translate of $u^*(x)$:

$$v(x,t) = pu^*_x(x) + \mathcal{O}(e^{-\delta t}) \qquad \Rightarrow \qquad u(x,t) \sim u^*(x) + pu^*_x(x) \sim u^*(x+p)$$

What if there is no spectral gap????

Pointwise semigroup estimates

Linear equation:

$$v_t = \mathcal{L}v \qquad \Rightarrow \qquad v(t) = \frac{1}{2\pi i}\int_\Gamma e^{\lambda t}(\lambda - \mathcal{L})^{-1} v_0 \, d\lambda = e^{\mathcal{L}t} v_0$$

Resolvent kernel: $G(x, y, \lambda)$

$$w(x) = \int_{\mathbb{R}} G(x, y, \lambda) v_0(y) dy \qquad \Rightarrow \qquad w = (\lambda - \mathcal{L})^{-1} v_0$$

Pointwise Green's function: $\mathcal{G}(x, y, t)$

$$v(x, t) = \int_{\mathbb{R}} \mathcal{G}(x, y, t) v_0(y) dy, \qquad \mathcal{G}(x, y, t) = \frac{1}{2\pi i}\int_\Gamma e^{\lambda t} G(x, y, \lambda) d\lambda$$

Example: $v_t = \mathcal{L}v = \partial_x^2 v$

$$G(x, y, \lambda) = \frac{1}{2\sqrt{\lambda}} e^{-\sqrt{\lambda}|x-y|}, \qquad \mathcal{G}(x, y, t) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{(x-y)^2}{4t}}.$$

In these kernels you can see:

- The spectrum $\sigma(\mathcal{L}) = (-\infty, 0]$
- Algebraic decay: $|\mathcal{G}(x, y, t)| \sim 1/\sqrt{t}$

Nonlinear stability via pointwise estimates

Solution to

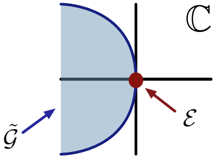
$$v_t = \mathcal{L}v + N(v)$$

is given by

$$v(x,t) = \int_{\mathbb{R}} \mathcal{G}(x,y,t)v_0(y)dy + \int_0^t \int_{\mathbb{R}} \mathcal{G}(x,y,t-s)N(v(s))dyds$$

Separate into decaying and non-decaying parts:

$$\mathcal{G}(x,y,t) = \mathcal{E}(x,y,t) + \tilde{\mathcal{G}}(x,y,t)$$



Developed by Zumbrun and colleagues (see eg [Zumbrun, Howard 98])

$$\begin{aligned} v(x,t) &= \int_{\mathbb{R}} \mathcal{E}(x,y,t)v_0(y)dy + \int_0^t \int_{\mathbb{R}} \mathcal{E}(x,y,t-s)N(v(s))dyds \\ &\quad + \int_{\mathbb{R}} \tilde{\mathcal{G}}(x,y,t)v_0(y)dy + \int_0^t \int_{\mathbb{R}} \tilde{\mathcal{G}}(x,y,t-s)N(v(s))dyds \end{aligned}$$

Nonlinear stability via pointwise estimates

Adjust the Ansatz

$$u_t = u_{xx} + f(u), \qquad u(x + p(t), t) = u^*(x) + v(x, t)$$

$$v_t = \mathcal{L}v + N(v) + \dot{p}[v_x + u^*_x]$$

Actively translate the solution via p; use to remove the neutral direction!

$$v(x, t) = \int_{\mathbb{R}} \mathcal{G} v_0 dy + \int_0^t \int_{\mathbb{R}} \mathcal{G}[N(v) + \dot{p} v_y] dy ds + (p(t) - p(0)) u^*_x(x).$$

Choose p to be defined implicitly as

$$p(t) u^*_x(x) = p(0) u^*_x(x) - \int_{\mathbb{R}} \mathcal{E} v_0(y) dy - \int_0^t \int_{\mathbb{R}} \mathcal{E}[N(v) + \dot{p} v_y] dy ds$$

so that v is governed by

$$v(x, t) = \int_{\mathbb{R}} \tilde{\mathcal{G}} v_0 dy + \int_0^t \int_{\mathbb{R}} \tilde{\mathcal{G}}[N(v) + \dot{p} v_y] dy ds \quad \Rightarrow \quad \text{decay!}$$

Wave translation captured by p equation - as if there were a spectral gap!

Sources

Sources are solutions to

$$u_t = Du_{xx} + f(u), \qquad x \in \mathbb{R}, \qquad u \in \mathbb{R}^n$$

Time-periodic in a moving frame

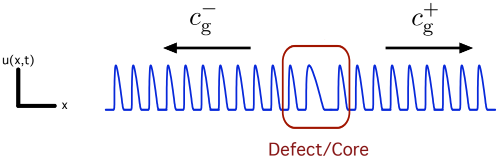
$$u^*(x,t), \qquad \tilde{x} = x - c^* t, \qquad WLOG \quad c^* = 0$$
$$u^*(x, t+2\pi) = u^*(x,t)$$

Spatially asymptotic to wave trains:

$$u_{\mathrm{wt}}(y;k) = u_{\mathrm{wt}}(kx - \omega(k)t;k) = u_{\mathrm{wt}}(y+2\pi;k)$$

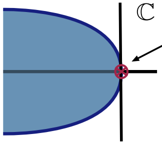
Group velocity:

$$c_{\mathrm{g}}(k) = \frac{d\omega}{dk}(k), \qquad c_{\mathrm{g}}^- < 0 < c_{\mathrm{g}}^+$$



Stability of sources

Linearize about source to get Floquet spectrum:



Double zero eigenvalue:

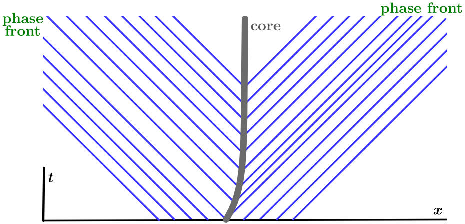
position: $u_x^*(x,t)$

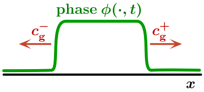
phase: $u_t^*(x,t)$

Difficulties for nonlinear stability:

- No spectral gap
- Linear operator is time-dependent: $\mathcal{L}(t)$
- Outward motion means perturbations don't stay localized

Anticipated effect of localized perturbation:





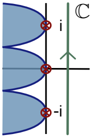
Time-periodicity of linear operator

Eigenvalue equation: $v_t = \mathcal{L}(t)v$ for $t \in (0, 2\pi)$, $v(2\pi) = e^{2\pi\sigma}v(0)$

Equivalently, $v(t) = e^{\sigma t}u(t)$, u is spatially localized and solves

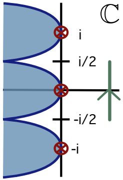
$$\sigma u + u_t = u_{xx} + f'(u^*(x,t))u, \quad u(x,t) = u(x, t+2\pi)$$

Two eigenfunctions for $\sigma = 0$: $\bar{u}_x(x,t)$ and $\bar{u}_t(x,t)$



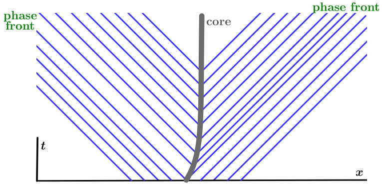
Green's function for periodic shocks [B., Sandstede, Zumbrun '10]:

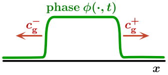
$$\mathcal{G}(x,t,y) = \frac{1}{2\pi i}\int_{\mu - i\infty}^{\mu + i\infty} e^{\lambda t} G(x,y,\lambda)d\lambda$$
$$= \frac{1}{2\pi i}\int_{\mu - \frac{i}{2}}^{\mu + \frac{i}{2}} e^{\sigma t} \sum_{n\in\mathbb{Z}} e^{int}\hat{G}^n(x,y,\sigma)d\sigma$$



Effect of phase fronts

Anticipated effect of localized perturbation:





Model equation:

$$\phi_t = \phi_{xx} - c\tanh\left(\frac{cx}{2}\right)\phi_x + \phi_x^2, \qquad c > 0.$$

Justification of model:

- Integrated Burgers captures phase dynamics [Howard and Kopell '77], [Doelman, Sandstede, Scheel, and Schneider '09]
- Advection terms represents group velocities

Mathematical difficulties

- No spectral gap; zero "eigenvalue" is constant function
- Localized initial data; solutions don't stay localized

Analysis of Model [B, Nguyen, Sandstede, Zumbrun '12]

Linear part:

$$\phi_t = \phi_{xx} - c \tanh\left(\frac{cx}{2}\right)\phi_x$$

Explicit Green's function ($\mathrm{erf}(z) = \int_{-\infty}^{z} e^{-s^2}\,ds$):

$$\mathcal{G}(x,y,t) = \underbrace{\left[\frac{1}{\sqrt{4\pi t}} e^{-\frac{(x-y-ct)^2}{4t}} \frac{1}{1+e^{-cy}} + \frac{1}{\sqrt{4\pi t}} e^{-\frac{(x-y+ct)^2}{4t}} \frac{1}{1+e^{+cy}}\right]}_{\text{Gaussians that decay algebraically}}$$

$$+\frac{c}{4}\underbrace{\left[\mathrm{erf}\left(\frac{y-x+ct}{\sqrt{4t}}\right) - \mathrm{erf}\left(\frac{y-x-ct}{\sqrt{4t}}\right)\right]}_{\text{outwardly spreading plateau}\to 1} \mathrm{sech}^2\left(\frac{cy}{2}\right)$$

Separate into bounded and decaying pieces:

$$\begin{aligned}
\mathcal{G}(x,y,t) &= \mathcal{E}(x,y,t) + \tilde{\mathcal{G}}(x,y,t), \\
\mathcal{E}(x,y,t) &= \frac{c}{4}\left[\mathrm{erf}\left(\frac{x+ct}{\sqrt{4t}}\right) - \mathrm{erf}\left(\frac{x-ct}{\sqrt{4t}}\right)\right]\mathrm{sech}^2\left(\frac{cy}{2}\right)
\end{aligned}$$

Ansatz must remove non-decaying part; plateau height instead of translation!

Analysis of Model [B, Nguyen, Sandstede, Zumbrun '12]

$$\phi_t = \phi_{xx} - c\tanh\left(\frac{cx}{2}\right)\phi_x + \phi_x^2$$

Ansatz construction:

- Haven't linearized about anything (ie no u^*)
- Use form of Green's function to "guess" a good Ansatz
- Can solve model exactly via Cole-Hopf

$$\phi(x,t) = \underbrace{\log(1+p(t)\mathcal{B}(x,t))}_{\phi^*(x,t,p(t))} + v(x,t), \qquad \mathcal{B}(x,t) = \mathcal{G}(x,0,t+1)$$

Note:

- $\phi^*(x,t,p_0)$ exact solution for any fixed p_0: nonlinear plateau, height p_0!
- Really only need ϕ^* to solve model with error $\leq \mathcal{O}((t+1)^{-1})$

Theorem: For each $\gamma \in (0, \frac{1}{2})$, $\exists\ \epsilon_0, \eta_0, C_0, M_0 > 0$ such that, if $\phi_0 \in C^1$ satisfies $\epsilon := \|e^{x^2/M_0}\phi_0\|_{C^1} \leq \epsilon_0$, then $\exists\ p_\infty$ with

$$|p(t) - p_\infty| \leq \epsilon C_0 e^{-\eta_0 t}, \qquad |v(x,t)| \leq \frac{\epsilon C_0}{(1+t)^\gamma}\left(e^{-\frac{(x+ct)^2}{M_0(t+1)}} + e^{-\frac{(x-ct)^2}{M_0(t+1)}}\right)$$

Sources in qCGL

Quintic complex Ginzburg-Laundau equation:

$$A_t = (1 + i\alpha)A_{xx} + A - (1 + i\beta)A|A|^2 + (\gamma_1 + i\gamma_2)A|A|^4$$

Sources are known to exist and have the form

$$A_{source}(x,t) = r(x)e^{i\varphi(x)}e^{-i\omega t} \to \pm r_0(k_0)e^{i[\pm k_0 x - \omega t]} \quad \text{as} \quad x \to \pm\infty$$

Asymptotic wave trains and group velocities:

$$\pm r_0(k_0)e^{\pm ik_0}e^{-i\omega(k_0)t}, \qquad c_g^{\pm} = \frac{d\omega}{dk}|_{\pm k_0} = \pm c_g, \qquad c_g^{-} > 0 > c_g^{+}$$

Two eigenfunctions

- $\partial_x A_{source}$: spatial translations
- $\partial_t A_{source}$: phase modulations

Two key aspects of qCGL vs general reaction diffusion equation

- Gauge invariance: $A \to Ae^{i\phi_0 t}$ allows one to remove the time-dependence
- Phase/amplitude coordinates: phase modulation is more transparent

Sources in qCGL

$$A_t = (1 + i\alpha)A_{xx} + A - (1 + i\beta)A|A|^2 + (\gamma_1 + i\gamma_2)A|A|^4$$

Stability Ansatz

$$A(x + p(x,t), t) = [r(x) + R(x,t)]e^{i[\varphi(x)+\phi(x,t)]}$$

Non-decaying part due to spatial translation:

- Removed by $p(x,t)$
- Spatially dependent because it is plateau-like (source - outward motion)

Non-decaying part due to phase modulation:

- Remove by considering an appropriate phase modulation

$$A_{mod}(x,t) = A_{source}(x,t)e^{i\phi^a(x,t)} = r(x)e^{i[\varphi(x)+\phi^a(x,t)]}$$

- Approximate phase modulation ϕ^a governed by a Burgers equation

$$\left(\partial_t + \frac{c_g}{k_0}\varphi_x\partial_x - d\partial_x^2\right)[\phi^a \pm k_0 p] = q[\partial_x(\phi^a \pm k_0 p)]^2$$

where d, q are related to the essential spectrum and nonlinearity, respectively.

Sources in qCGL

Theorem [B., Nguyen, Sandstede, Zumbrun '14]: Assume that the initial data is of the form $A_{\mathrm{in}}(x) = R_{\mathrm{in}}(x)e^{i\phi_{\mathrm{in}}(x)}$ and A_{source} is spectrally stable. There exists a positive constant ϵ_0 such that, if

$$\epsilon := \|e^{x^2/M_0}(R_{\mathrm{in}} - r)(\cdot)\|_{C^3(\mathbb{R})} + \|e^{x^2/M_0}(\phi_{\mathrm{in}} - \varphi)(\cdot)\|_{C^3(\mathbb{R})} \le \epsilon_0,$$

then the solution $A(x,t)$ to the qCGL equation exists globally in time. In addition, there are constants $\eta_0, C_0, M_0 > 0$ and appropriate solutions of Burgers equation, p and ϕ^a, so that

$$\begin{aligned}&\Big|\frac{\partial^\ell}{\partial x^\ell}\Big[A(x+p(x,t),t) - A_{\mathrm{mod}}(x,t)\Big]\Big| \\ &\qquad \le \epsilon C_0(1+t)^{\kappa-1/2}[(1+t)^{-\ell/2} + e^{-\eta_0|x|}]\left(e^{-\frac{(x-c_g t)^2}{M_0(t+1)}} + e^{-\frac{(x+c_g t)^2}{M_0(t+1)}}\right)\end{aligned}$$

for $x \in \mathbb{R}$, $t \ge 0$, and $\ell = 0,1,2$ and for each fixed $\kappa \in (0, \frac{1}{2})$. In particular, $\|A(\cdot + p(\cdot,t),t) - A_{\mathrm{mod}}(\cdot,t)\|_{W^{2,r}} \to 0$ as $t \to \infty$ for each fixed $r > \frac{1}{1-2\kappa}$.

Explanation of Ansatz

Two key mathematical difficulties:

- (1) Removal of non-decaying terms due to embedded zero eigenvalues
- (2) Dealing with quadratic nonlinearity

(1): resolved by now standard but nontrivial pointwise semigroup methods

(2) Even if $\mathcal{G}$ decayed like a Gaussian, quadratic terms can have a nontrivial effect on the dynamics. Consider, eg,

$$u_t = u_{xx} - u^2$$

Non-positive initial data lead to solutions that don't decay to zero!

Ansatz involving functions that solve Burgers equation removes not only the non-decaying pieces but also the leading order quadratic nonlinear terms, thus allowing us to close our nonlinear estimates.

Explanation of Ansatz

$$A_t = (1 + i\alpha)A_{xx} + A - (1 + i\beta)A|A|^2 + (\gamma_1 + i\gamma_2)A|A|^4$$

Where does the Burgers equation come from?

$$A(x + p(x,t), t) = (r(x) + R(x,t))e^{i[\varphi(x)+\phi(x,t)]}$$

Perturbation satisfies

$$\begin{aligned}
\partial_t \begin{pmatrix} R \\ r\phi \end{pmatrix} \quad \approx \quad & \begin{pmatrix} 1 - \alpha\beta^* & \alpha r \\ -\alpha(1 + \beta^*) & r(1 + \alpha\beta^*) \end{pmatrix} \partial_x^2 \begin{pmatrix} R \\ \phi \end{pmatrix} \\
& + \begin{pmatrix} -2(\alpha + \beta^*)\varphi_x & 2r\varphi_x \\ -2\varphi_x(1 + (\beta^*)^2) & 2r\varphi_x(\beta^* - \alpha) \end{pmatrix} \partial_x \begin{pmatrix} R \\ \phi \end{pmatrix} \\
& + \begin{pmatrix} -2r^2(1 - 2\gamma_1 r^2) & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} R \\ \phi \end{pmatrix} + \begin{pmatrix} \mathcal{O}(R^2, \phi_x^2, R\phi_x) \\ qr\phi_x^2 \end{pmatrix},
\end{aligned}$$

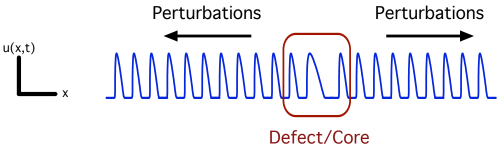
where $c_g = 2k_0(\alpha - \beta^*)$, $\beta^* = \beta^*(r_0, \gamma)$, and $q = q(\alpha, \beta, k_0, \gamma)$.

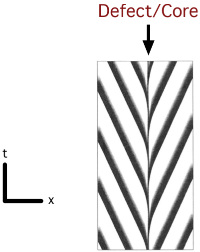
Notice:

- R decays faster than ϕ.
- ϕ governed to leading order by Burgers equation.

Summary

GOAL: Understand nonlinear stability of sources:





Difficulties at the linear level:

- Embedded zero eigenvalues lead to nondecaying semigroup
- Pointwise Green's function allows for separation of the nondecaying part, similar to the situation with a spectral gap via a spectral projection
- Source creates outward motion of perturbations - perturbations don't stay localized

Difficulties at the nonlinear level:

- Quadratic nonlinearities potentially problematic
- Ansatz must remove leading order nonlinear terms

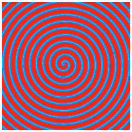
Summary

Resolution:

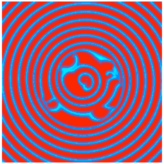
- Use now-standard pointwise methods to deal with linear difficulties
- Additional estimates at linear level needed to deal with outward motion
- Ansatz via Burgers equation to remove quadratic nonlinear terms

Future directions:

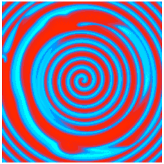
- Sources in general reaction-diffusion equations
- Sources in higher spatial dimensions - spiral waves



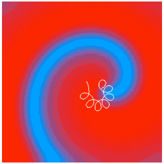
Spirals can undergo an interesting variety of instabilities and bifurcations



Core break-up



Far-field break up



Tip meander

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