Nonlinear stability of semidiscrete shocks for two-sided schemes

Margaret Beck
Boston University

Joint work with
Hermen Jan Hupkes, Björn Sandstede, and Kevin Zumbrun

Setting: semi-discrete conservation laws

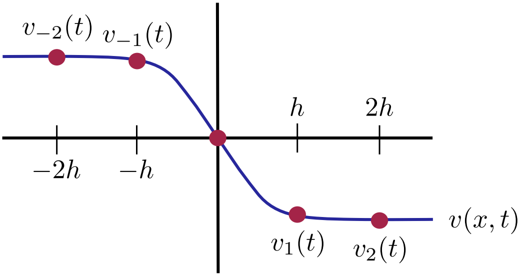
Semi-discrete conservation law: $j \in \mathbb{Z}$, $v_j(t) \in \mathbb{R}^N$

$$\frac{dv_j}{dt} + \frac{1}{h}[f(v_{j-p+1}, \dots, v_{j+q}) - f(v_{j-p}, \dots, v_{j+q-1})] = 0$$

Discretization of: $x \in \mathbb{R}$, $v(x,t) \in \mathbb{R}^N$

$$v_t + \bar{f}(v)_x = 0, \qquad \bar{f}(u) = f(u, \dots, u)$$

- Spatial step size h
- Size of discretization stencil: p, q



Traveling waves

$$\frac{dv_j}{dt} + \frac{1}{h}[f(v_{j-p+1}, \dots, v_{j+q}) - f(v_{j-p}, \dots, v_{j+q-1})] = 0$$

Traveling wave: speed σ

$$v_j(t) = u_*\left(j - \frac{\sigma}{h}t\right), \quad u_* : \mathbb{R} \to \mathbb{R}^N$$

Traveling wave equation: $x = j - \sigma t/h$

$$\sigma u_*'(x) = f(u_*(x-p+1), \dots, u_*(x+q)) - f(u_*(x-p), \dots, u_*(x+q-1))$$

- FDE of mixed type
- Dynamics depend on past (p) and future (q)

We will assume:

- $\lim_{x\to\pm\infty} u_*(x) = u_\pm$
- u_* is a Lax shock: (1) ordered eigenvalues $a_1^\pm < \cdots < a_N^\pm$ of $\bar{f}_u(u_\pm)$;
  (2) $a_{k-1}^- < \sigma < a_k^-$; $a_k^+ < \sigma < a_{k+1}^+$
- $u_*'(x)$ does not vanish on any interval of length at least $p+q$

Ill-posed equation

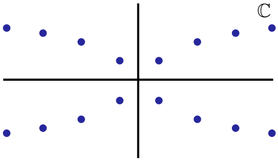
FDEs of mixed type are generally ill-posed: ($p = q = 1$)

$$u'(x) = u(x+1) + u(x-1), \quad u_0(x)|_{[-1,1]} = 1$$

Related to spectrum:

$$u(x) = e^{\lambda x} \quad \Rightarrow \quad \lambda = e^{\lambda} + e^{-\lambda}$$

- Solutions with $\mathrm{Re}\lambda \to \pm\infty$
- Solutions grow arbitrarily fast in forwards and backwards "time"
- No semiflow



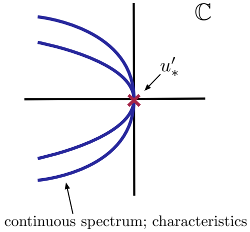
Solve with exponential dichotomies: [Härterich, Sandstede, Scheel 02]
[Mallet-Paret, Verduyn-Lunel, Preprint]

Candidates for nonlinear stability

GOAL: given existence and spectral stability, prove nonlinear stability

$\Rightarrow$ Spectral stability: roughly speaking

- No spectrum in right half plane
- Minimal multiplicity zero eigenvalue: u'_*
- Continuous spectrum is parabolic, generated solely by characteristics



$\Rightarrow$ Our results will apply to weak shocks: $|u_+ - u_-| \ll 1$

- Existence
  - Upwind [Benzoni-Gavage 98], mixed [Benzoni-Gavage, Huot 02]
  - Constructed using a center manifold
- Spectral stability
  - Essentially in [Benzoni-Gavage, Huot, Rousset 03]
  - No spectrum in unstable half-pane: energy estimates
  - Minimal multiplicity of zero eigenvalue: from existence construction

Spectral stability of TW

Semi-discrete equation in moving coordinate:

$$\frac{d}{dt}v = \sigma v'(x) - f(v(x-p+1),\ldots,v(x+q)) + f(v(x-p),\ldots,v(x+q-1))$$

Linearized operator: $\mathcal{L} : L^2(\mathbb{R},\mathbb{C}^N) \to L^2(\mathbb{R},\mathbb{C}^N)$

$$\begin{aligned}
(\mathcal{L}u)(x) &= \sigma u'(x) - \sum_{j=-p}^{q} A_j(x)u(x+j) \\
A_j(x) &= \partial_j f(u_*(x-p+1),\ldots,u_*(x+q)) \\
&\quad -\partial_{j+1} f(u_*(x-p),\ldots,u_*(x+q-1))
\end{aligned}$$

Floquet-like spectral structure:

$$\mathcal{L}(e^{2\pi \mathrm{i} x}u) = e^{2\pi \mathrm{i} x}(2\pi\mathrm{i}\sigma + \mathcal{L})u$$

- Spectrum invariant under shifts by $2\pi\mathrm{i}\sigma$
- Lattice doesn't feel oscillations on scale small than distance in lattice

Linear Evolution: two perspectives

(1) Moving coordinate frame:

$$\partial_t v = \mathcal{L} v, \quad v(t) \in L^2(\mathbb{R}, \mathbb{C}^N)$$

(2) Coordinates of Lattice:

$$\partial_t v_j = F(v)_j, \quad v_j^*(t) = u_*\left(j - \frac{\sigma}{h} t\right), \quad v_j^*\left(t + \frac{h}{\sigma}\right) = v_{j-1}(t)$$

- TW is relative periodic orbit
- Linearization:

$$\partial_t v = F_v(v^*(t))v)$$

$\Rightarrow$ For analysis it's convenient to

- Use (1) for spectral assumptions and resolvent kernel construction
- Use (2) for pointwise Green's function and nonlinear analysis

Linear Evolution: connecting the two perspectives

(1) Moving coordinate frame:

$$\partial_t v = \mathcal{L} v, \quad (\mathcal{L} - \lambda) G = \delta(\cdot - y)$$

- Spectral assumptions on $\mathcal{L}$
- Resolvent kernel $G(x, y, \lambda)$

(2) Coordinates of Lattice:

$$\partial_t v = (F_v(v^*(t))v) + N(v), \qquad v(t) = \{v_j(t)\}_{j \in \mathbb{Z}}$$

- Green's function

$$v_j(t) = \sum_{i \in \mathbb{Z}} \mathcal{G}(j, i, t, 0) v_i(0) + \int_0^t \sum_{i \in \mathbb{Z}} \mathcal{G}(j, i, t, s) N(v(s))_i ds$$

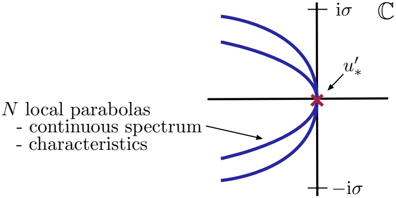
- Relationship [Benzoni-Gavage, Huot, Rousset '03]

$$\mathcal{G}(j, i, t, s) = \mathcal{G}\left(j - \frac{\sigma}{h} t, i - \frac{\sigma}{h} s, t - s\right)$$

$$\mathcal{G}(x, y, \tau) = -\frac{1}{2\pi \mathrm{i}} \int_{\gamma - \mathrm{i}\pi\sigma}^{\gamma + \mathrm{i}\pi\sigma} e^{\lambda\tau} G(x, y, \lambda) d\lambda$$

General strategy for nonlinear stability

- Assume: existence, spectrum of $\mathcal{L}$ in moving frame



- Construct resolvent kernel

$$(\mathcal{L} - \lambda)G = \delta(\cdot - y) \quad \Rightarrow \quad G(x, y, \lambda)$$

- Connect to behavior on lattice

$$\mathcal{G}(j, i, t, s) = \mathcal{G}\left(j - \frac{\sigma}{h}t, i - \frac{\sigma}{h}s, t - s\right)$$
$$\mathcal{G}(x, y, \tau) = -\frac{1}{2\pi \mathrm{i}} \int_{\gamma - \mathrm{i}\pi\sigma}^{\gamma + \mathrm{i}\pi\sigma} e^{\lambda\tau} G(x, y, \lambda) d\lambda$$

- Determine pointwise bounds on $\mathcal{G}$ sufficient for nonlinear stability.

Previous results and new challenges

- Upwind schemes: nonlinear stability proven
  - Under assumptions similar to above [Benzoni-Gavage, Huot, Rousset 03]
  - Spectrum unbounded in one direction; resolvent kernel via Evans function
- Mixed type schemes:
  - Can't use Evans function for resolvent kernel
  - Ill-posed equation requires exponential dichotomies
  - Dichotomies constructed [Härterich, Sandstede, Scheel 02] [Mallet-Paret, Verduyn-Lunel, Preprint]
- Related analysis:
  - Similar analysis for time-periodic viscous shocks [B., Snadstede, Zumbrun 10]
  - Pointwise Greens function estimates [Zumbrun and colleagues]
- New analysis required:
  - Construct resolvent kernel using dichotomy and get pointwise bounds
  - Relationship between dichotomy for $\mathcal{L}$ and its adjoint
  - Regularity of dichotomy in y

Exponential dichotomies

Recall setting for non-discrete case:

$$\lambda u = \underbrace{u_{xx} + a(x)u_x + b(x)u}_{\mathcal{L}u}, \quad a(x) \to a_\pm, \quad b(x) \to b_\pm$$

Spatial dynamical system

$$\frac{d}{dx}\begin{pmatrix} u \\ v \end{pmatrix} = \mathcal{A}(x;\lambda)\begin{pmatrix} u \\ v \end{pmatrix}, \quad \mathcal{A}(x;\lambda) = \begin{pmatrix} 0 & 1 \\ \lambda - b(x) & -a(x) \end{pmatrix}$$

Solve on $L^2(\mathbb{R}, \mathbb{R}^{2N})$ using dichotomy:

$$\frac{d}{dx}\begin{pmatrix} u \\ v \end{pmatrix} = \mathcal{A}(x;\lambda)\begin{pmatrix} u \\ v \end{pmatrix} + \mathcal{F}$$

$$\begin{pmatrix} u \\ v \end{pmatrix}(x) = \int_{-\infty}^{x} \Phi^s(x,z,\lambda)\mathcal{F}(z)dz + \int_{+\infty}^{x} \Phi^u(x,z,\lambda)\mathcal{F}(z)dz$$

Resolvent kernel comes from dichotomy:

$$G(x,y,\lambda) = \begin{cases} \int_{-\infty}^{x} \Phi_1^s(x,z,\lambda)\mathcal{F}(z)dz & x > y \\ -\int_{+\infty}^{x} \Phi_1^u(x,z,\lambda)\mathcal{F}(z)dz & x < y \end{cases}, \qquad \mathcal{F}(z) = \begin{pmatrix} 0 \\ \delta(z-y) \end{pmatrix}$$

Exponential dichotomies

Semi-discrete case:

$$\lambda u(x) = \underbrace{\sigma u'(x) - \sum_{j=-p}^{q} A_j(x) u(x+j)}_{\mathcal{L}u}, \quad A_j(x) \to A_j^{\pm}$$

To get spatial dynamical system:

- Already first order in x
- Need to deal with advance/delay

$$\frac{d}{dx}\begin{pmatrix}\phi \\ \alpha\end{pmatrix} = \mathcal{A}(x;\lambda)\begin{pmatrix}\phi \\ \alpha\end{pmatrix} = \begin{pmatrix}\phi_z \\ \frac{1}{\sigma}(A_0(x)-\lambda)\alpha + \frac{1}{\sigma}\sum_{j=-p,j\neq 0}^{q} A_j(x)\phi(j)\end{pmatrix} \quad (*)$$

Think of:

$$\begin{aligned} u(x) \in L^2(\mathbb{R}, \mathbb{C}^N) &\quad \to \quad \text{eigenfunction} \\ \phi(x,z) = u(x+z) \in L^2([-p,q], \mathbb{C}^N) &\quad \to \quad \text{local piece of } u \text{ at } x \\ \alpha(x) = \phi(x,0) \in \mathbb{C}^N &\quad \to \quad \text{eigenfunction} \end{aligned}$$

Solve $(*)$ on $L^2(\mathbb{R}, Y)$, $Y := L^2([-p,q], \mathbb{C}^N) \times \mathbb{C}^N$

Resolvent kernel bounds

To get resolvent kernel:

- Construct dichotomy using [Härtereich, Sandstede, Scheel 02] for

$$\frac{d}{dx}\begin{pmatrix}\phi\\ \alpha\end{pmatrix} = \mathcal{A}(x;\lambda)\begin{pmatrix}\phi\\ \alpha\end{pmatrix}$$

- Meromorphic extension of $\Phi^{s,u}$ into neighborhood of $\lambda = 0$.
- Using spectral assumptions and adjoint relationship find

$$G(x,y,\lambda) = \frac{1}{\lambda} u_*'(x) \sum_{\mathrm{in}} e^{-\nu_{\mathrm{in}}^{\pm}(\lambda)} P_{\mathrm{in}} + \tilde{G}(x,y,\lambda)$$

where

$$\frac{1}{\lambda} u_*'(x) \sum_{\mathrm{in}} e^{-\nu_{\mathrm{in}}^{\pm}(\lambda)} P_{\mathrm{in}} \quad \rightarrow \quad \text{records effects leading to translation}$$

$$\tilde{G}(x,y,\lambda) \quad \rightarrow \quad \text{higher order terms, including effects of characteristics/continuous spectrum}$$

Technical issues: relationship involving adjoint; regularity of dichotomies

Pointwise Green's function

Use relationship between lattice and moving coordinate:

$$\mathcal{G}(j,i,t,s) = \mathcal{G}\left(j - \frac{\sigma}{h}t, i - \frac{\sigma}{h}s, t-s\right)$$

$$\begin{aligned}
\mathcal{G}(x,y,\tau) &= -\frac{1}{2\pi\mathrm{i}}\int_{\gamma-\mathrm{i}\pi\sigma}^{\gamma+\mathrm{i}\pi\sigma} e^{\lambda\tau}\left[\frac{1}{\lambda}u_*'(x)\sum_{\mathrm{in}} e^{-\nu_{\mathrm{in}}^{\pm}(\lambda)}P_{\mathrm{in}} + \tilde{G}(x,y,\lambda)\right]d\lambda \\
&= \mathcal{E}(x,y,\tau) + \tilde{\mathcal{G}}(x,y,\tau)
\end{aligned}$$

Pieces of Greens function:

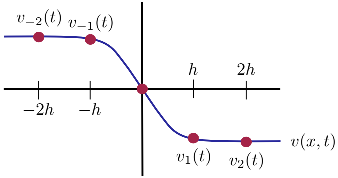
$$\mathcal{E}(x,y,\tau) = u_*'(x)\sum_{\mathrm{in}}\left[\mathrm{errfn}\left(\frac{x+a_{\mathrm{in}}\tau}{\sqrt{4(\tau+1)}}\right) - \mathrm{errfn}\left(\frac{x-a_{\mathrm{in}}\tau}{\sqrt{4(\tau+1)}}\right)\right]$$

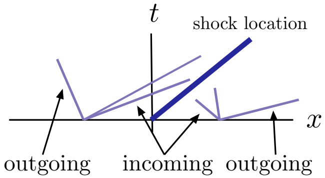
records translation due to neutral eigenvalue

$\tilde{\mathcal{G}}(x,y,\tau)$ = Gaussians moving along outgoing characteristics

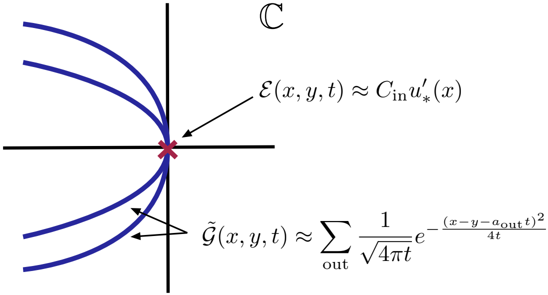
Pointwise Green's function

Shock and characteristics:





Spectrum and Green's function:



Remark: equation is hyperbolic, discretization induces dissipation

Nonlinear stability

Original semi-discrete equation:

$$\frac{dv_j}{dt} + \frac{1}{h}[f(v_{j-p+1}, \ldots, v_{j+q}) - f(v_{j-p}, \ldots, v_{j+q-1})] = 0$$

Solution Ansatz:

$$v(t+\rho(t)) = v_*(t) + u(t), \quad u = \text{ perturbation}, \quad \rho = \text{ phaseshift}$$

Perturbation satisfies

$$w_j(t) = \sum_{i\in\mathbb{Z}} \mathcal{G}(j,i,t,0)w_i(0) + \int_0^t \sum_{i\in\mathbb{Z}} \mathcal{G}(j,i,t,s)N(w(s))_i ds + v'_{*,i}(t)\rho(t)ds$$

Chose ρ to cancel nondecaying $\mathcal{E}$:

$$w_j(t) = \sum_{i\in\mathbb{Z}} \tilde{\mathcal{G}}(j,i,t,0)w_i(0) + \int_0^t \sum_{i\in\mathbb{Z}} \tilde{\mathcal{G}}(j,i,t,s)N(w(s))_i ds$$

Theorem

A nondegenerate Lax shock that satisfies the spectral stability conditions is nonlinearly stable as follows. There is an $\epsilon, K > 0$ so that for each $\{v_j(0)\}_{j\in\mathbb{Z}}$ with $|\{v_j(0) - u_*(j)\}_{j\in\mathbb{Z}}|_{L^1} \le \epsilon$, there is a $\rho : \mathbb{R}^+ \to \mathbb{R}$ with

$$\sup_{t\ge 0}\Big(|\rho(t)| + (1+|t|)^{\frac{1}{2}}|\dot{\rho}(t)|\Big) \le K\epsilon$$

such that the solution $\{v_j(t)\}_{j\in\mathbb{Z}}$ satisfies

$$|\{v_j(t) - u_*(j+\rho(t)-\sigma t/h)\}_{j\in\mathbb{Z}}|_{L^\alpha(\mathbb{Z})} \le \frac{K\epsilon}{(1+|t|)^{\frac{1}{2}(1-\frac{1}{\alpha})}}, \qquad t \ge 0$$

for each $\alpha \ge 1$, where

$$L^\alpha(\mathbb{Z}) := \left\{ v : \mathbb{Z} \to \mathbb{R}^N : \ |v|_{L^\alpha(\mathbb{Z})} := \Big[\sum_{j\in\mathbb{Z}} |v_j|^\alpha\Big]^{1/\alpha} < \infty \right\},$$

If the initial condition satisfies $|v_j(0) - u_*(j)| \le \epsilon(1+jh)^{-3/2}$ for $j \in \mathbb{Z}$, then there exists a ρ_∞ with $|\rho_\infty| \le K\epsilon$ for which

$$|\rho(t) - \rho_\infty|\,(1+|t|)^{1/2} + |\dot{\rho}(t)|\,(1+|t|) \le K\epsilon.$$

Summary

Studied Lax shock solutions of

$$\frac{dv_j}{dt} + \frac{1}{h}[f(v_{j-p+1}, \dots, v_{j+q}) - f(v_{j-p}, \dots, v_{j+q-1})] = 0$$

Proved nonlinear stability under assumptions: (eg weak Lax shocks)

- Existence and nondegeneracy
- Spectral stability

Key aspects of analysis

- Resolvent kernel constructed with dichotomy, not Evans function
- Technical issues resolved: regularity, adjoints

Spectral stability assumptions

Assumptions on $\mathcal{L}$ with domain $H^1(\mathbb{R}, \mathbb{C}^N)$:

(S1) No spectrum in $\{\mathrm{Re}\lambda \geq 0\} \setminus 2\pi \mathrm{i}\sigma\mathbb{Z}$
(S2) u_*' only solution of $\mathcal{L}u = 0$
(S3) Minimal multiplicity: $\det[r_1^-, \ldots, r_{k-1}^-, [u_+ - u_-], r_{k+1}^+, \ldots, r_N^+] \neq 0$
(S4) Nonresonance: for $\xi \in \mathbb{R} \setminus \{0\}$,

$$\det\left[\mathrm{i}\sigma\xi - \sum_{j=-p}^{q} (\partial_j - \partial_{j+1}) f(u_\pm, \ldots, u_\pm) e^{\mathrm{i}\xi j}\right] \neq 0$$

(S5) Scheme is dissipative: $\langle l_n^\pm, B^\pm r_n^\pm \rangle > 0, \quad n = 1, \ldots, N$

Notation:

- Left/right eigenvectors of $\bar{f}_u(u_\pm)$: $r_n^\pm$, $l_n^\pm$
- Outgoing characteristic if eigenvalue:

$$a_1^-, \ldots, a_{k-1}^- < \sigma < a_{k+1}^+, \ldots, a_N^+$$

- Viscosity matrix:

$$B^\pm = (1/2) \sum_{j=-p}^{q} (1 - 2j) \partial_j f(u_\pm, \ldots, u_\pm)$$