Snakes, ladders, and isolas of localised patterns

Margaret Beck
University of Surrey

Jürgen Knobloch
Technical University Ilmenau

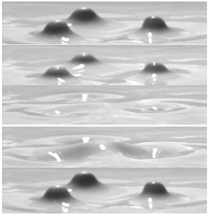
David Lloyd
University of Surrey

Björn Sandstede
University of Surrey

Thomas Wagenknecht
University of Leeds

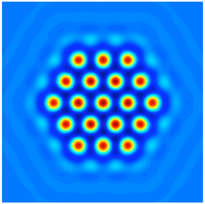
Localised Patterns

Experiments: Oscillons in Colloidal Suspension



[Lioubashevski et al. 99]

Numerics: Stationary solutions of 2D Swift-Hohenberg Equation

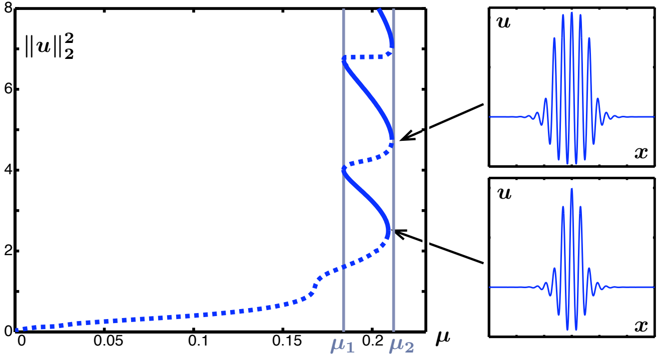


[Lloyd, Sandstede, Avitabile, Champneys 08]

Motivation

- GOAL: Determine sufficient conditions that guarantee "snaking" in the formation of localized patterns

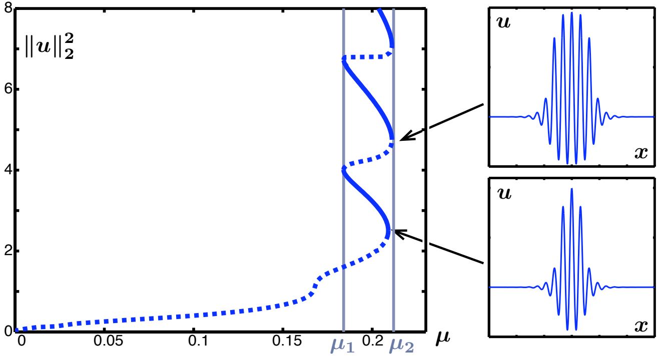
$$U_t = -(1+\partial_x^2)^2 U - \mu U + \nu U^2 - U^3$$



Motivation

- GOAL: Determine sufficient conditions that guarantee "snaking" in the formation of localized patterns

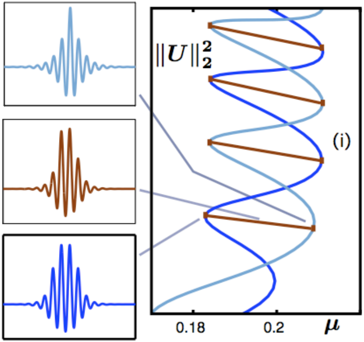
$$0 = -(1 + \partial_x^2)^2 U - \mu U + \nu U^2 - U^3$$



Motivation

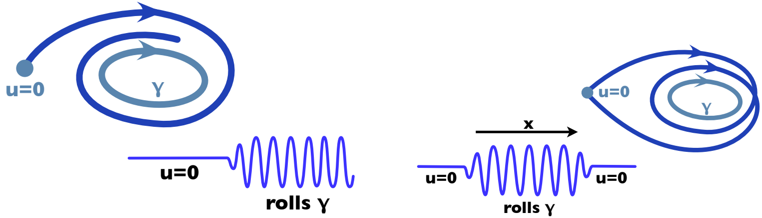
- GOAL: Determine sufficient conditions that guarantee "snaking" in the formation of localized patterns

$$0 = -(1 + \partial_x^2)^2 U - \mu U + \nu U^2 - U^3$$



Fronts vs Localised Patterns

One way to view localised patterns:



- Heteroclinic connection: front
- Homoclinic orbit: localised pattern

Main idea: Assume we know bifurcation diagram for fronts; use it to determine that of localised patterns

Set-up in 1D

Main example: Swift-Hohenberg

$$0 = -(1 + \partial_x^2)^2 U - \mu U + \nu U^2 - U^3$$

Stationary solutions solve first-order ODE: $u = (U, U_x, U_{xx}, U_{xxx})$

$$u_x = f(u, \mu), \quad u(x) \in \mathbb{R}^4$$

Key properties:

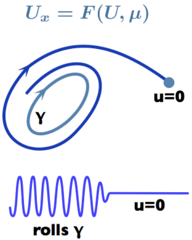
- Conservation of energy: $\exists\ \mathcal{H}$ such that

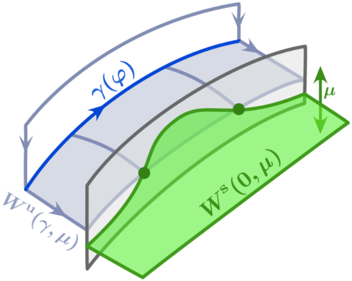
$$\frac{d}{dx}\mathcal{H}(u(x), \mu) = 0$$

- Reversibility: $x \to -x$
- Background state: $u(x) \equiv 0$, $f(0, \mu) = 0$, $\mathcal{H}(0, \mu) = 0$
- Periodic orbits: $\gamma(x, \mu)$, symmetric, $\mathcal{H}(\gamma, \mu) = 0$

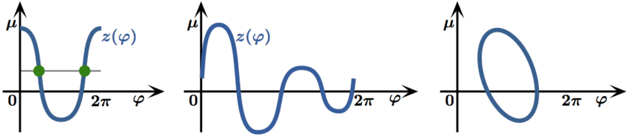
Description of Results

Assume bifurcation structure of fronts is known:



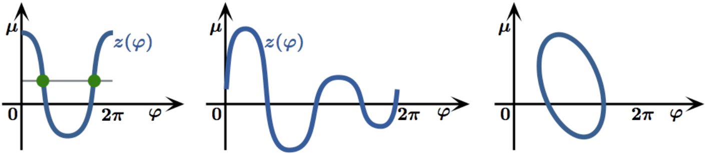


Hypothesis: Fronts exist for $\mu = z(\varphi)$ for an appropriate function z.

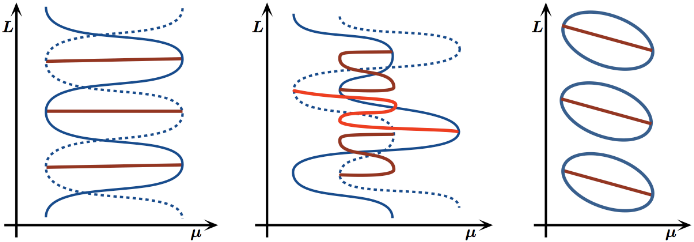


Description of Results

Bifurcation structure of fronts determines that of localised patterns:



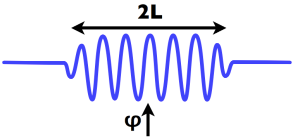
Leads to snaking curves:



Description of Results

How to see this:

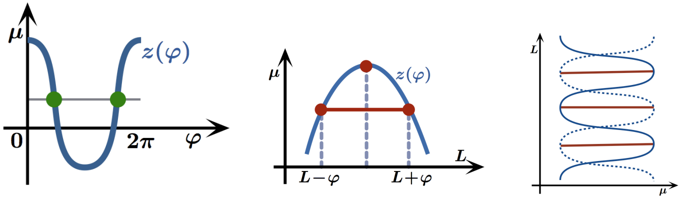
- Bifurcation equations:



Symmetric: $\mu = z(L+\varphi) + \mathcal{O}(e^{-\eta L})$

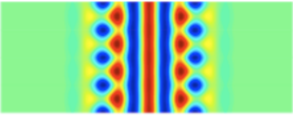
Asymmetric: $z(L+\varphi) = z(L-\varphi)$

- Use to trace out snaking curve

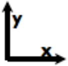


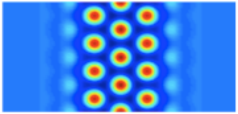
2D Patterns

Analysis of 2D patterns localised in x, periodic in y:



roll-square structures





hexagon pulses

$$0 = -(1+\triangle)^2 U - \mu U + \nu U^2 - U^3 \quad \to \quad u_x = f(u,\mu)$$

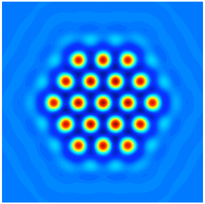
Infinite dimensional dynamical system:

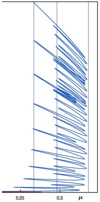
$$u(x) = (U, U_x, U_{xx}, U_{xxx})(x) \in H^3(S^1) \times H^2(S^1) \times H^1(S^1) \times L^2(S^1)$$

For each x, u is periodic function of y.

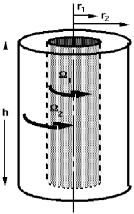
Open problems and future directions

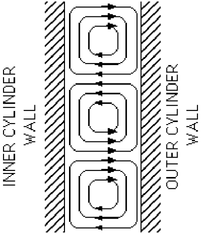
- Stability of localised patterns: Numerics [Burke, E Knobloch]
- Analysis of 2D patches:





- Boundary effects in bifurcations: Taylor vorticies





[R Tagg, CU Denver]

[A Weisberg, Princeton]