Nonlinear Stability of Time-periodic Viscous Shocks

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Abstract

Time-periodic shocks in systems of viscous conservation laws are shown to be nonlinearly stable. The result is obtained by representing the evolution associated to the linearized, time-periodic operator using a contour integral, similar to that of strongly continuous semigroups. This yields detailed pointwise estimates on the Green's function for the time-periodic operator. The evolution associated to the embedded zero eigenvalues is then extracted. Stability follows from a Gronwall-type estimate, proving algebraic decay of perturbations.

Introduction

Consider a parabolic system of conservation laws

$$u_t + F(u)_x = u_{xx}, \quad x \in \mathbb{R}, \quad u \in \mathbb{R}^n. \tag{1}$$

Recently, it has been shown that time-periodic shocks of the form

$$u(x,t) = \bar{u}^{\mathrm{per}}(x-ct,t), \quad \lim_{\xi\to\pm\infty} \bar{u}^{\mathrm{per}}(\xi,t) = u_\pm, \quad \bar{u}^{\mathrm{per}}(\xi,t+2\pi) = \bar{u}^{\mathrm{per}}(\xi,t),$$

can result from Hopf bifurcations of stationary shocks [TZ08, SS08]. Physically, their study is motivated by detonation waves. We focus on Lax shocks, but the results can be extended to over-, under-compressive, and mixed type waves.

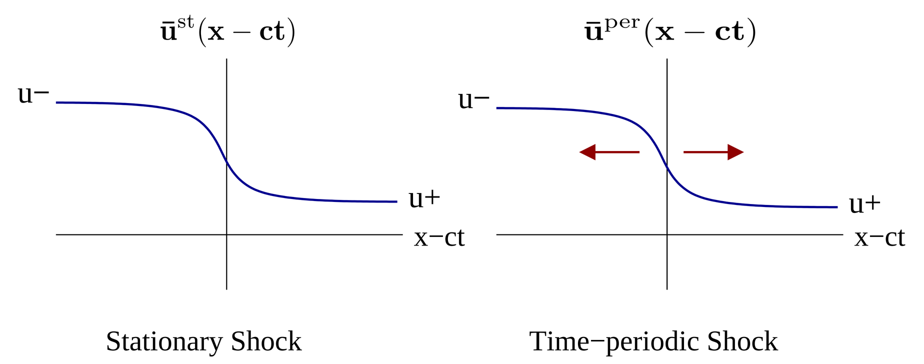


Figure 1: Both stationary and time-periodic viscous shocks, shown in a co-moving frame, have asymptotic limits $u_\pm$. On the right, the interior can vary periodically.

GOAL: Prove, under appropriate spectral stability assumptions, that solutions of the form $\bar{u}^{\mathrm{per}}$ are nonlinearly stable.

Assume, WLOG, that $c = 0$. Linearization about $\bar{u}^{\mathrm{per}}$ shows that perturbations satisfy

$$v_t = \mathcal{L}(t)v + Q(v,v_x)_x, \quad \mathcal{L}(t)v = v_{xx} - (F_u(\bar{u}^{\mathrm{per}}(x,t))v)_x. \tag{2}$$

Mathematical challenges:

- Time-dependent linear operator; no standard spectral or semigroup theory
- No spectral gap: two zero eigenvalues embedded in continuous floquet spectrum

Strategy:

- Develop contour integral representation of linear evolution, similar to semigroups
- Define time-periodic Green's function and obtain pointwise estimates
- Prove nonlinear stability via integral representation of solutions

Background: stationary shocks

Consider first stationary shocks of the form

$$u(x,t) = \bar{u}^{\mathrm{st}}(x-ct), \quad \lim_{\xi\to\pm\infty} \bar{u}^{\mathrm{st}}(\xi) = u_\pm.$$

Assume, WLOG, that $c = 0$. Linearization about $\bar{u}^{\mathrm{st}}$ shows that perturbations satisfy

$$v_t = \mathcal{L}v + Q(v,v_x)_x, \quad \mathcal{L}v = v_{xx} - (F_u(\bar{u}^{\mathrm{st}}(x))v)_x.$$

Assume the spectrum is as in figure 2, with simple zero eigenvalue and eigenfunction $\bar{u}^{\mathrm{st}}_x$. The linear evolution can be understood by using the Laplace transform:

$$\hat{v}(x,\lambda) = \int_0^\infty e^{-\lambda t} v(x,t)\,dt$$

$$v_t = \mathcal{L}v \quad \Longrightarrow \quad \lambda\hat{v} - v_0 = \mathcal{L}\hat{v}$$

Solve with the resolvent operator and invert the Laplace transform to obtain the standard contour integral representation of the semigroup:

$$v(t) = e^{t\mathcal{L}}v_0 = \frac{1}{2\pi i}\int_\Gamma e^{t\lambda}(\lambda - \mathcal{L})^{-1}v_0\,d\lambda.$$

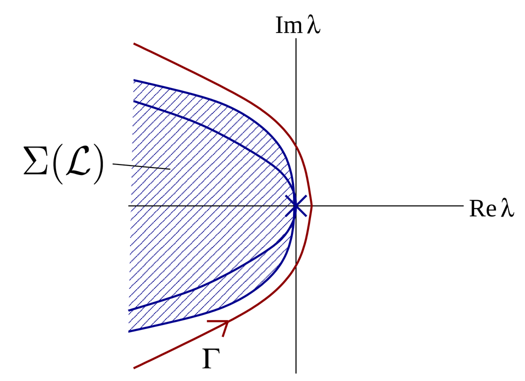


Figure 2: Spectrum Σ for stationary shocks and the contour Γ, used in (4).

The resolvent kernel, $G(x,y,\lambda)$, satisfies

$$\lambda G - \delta(\cdot - y) = \mathcal{L}G, \quad ((\lambda - \mathcal{L})^{-1}v_0)(x) = \int_{\mathbb{R}} G(x,y,\lambda)v_0(y)dy \tag{3}$$

and leads to the pointwise Green's function

$$\mathcal{G}(x,t,y) = \frac{1}{2\pi i}\int_\Gamma e^{t\lambda}G(x,y,\lambda)d\lambda. \tag{4}$$

By expanding $G(x,y,\lambda)$ near $\lambda = 0$, this representation has been used to obtain large-time asymptotics for $\mathcal{G}(x,t,y)$ and prove stability of stationary shocks [HZ06].

Contour integral representation of the linear evolution

To study time-periodic shocks, we seek a representation similar to (4). Because $\mathcal{L}(t)$ is time-periodic, we use the Floquet spectrum, defined as

$$\Sigma = \{\sigma \in \mathbb{C} : e^{2\pi\sigma} \in \text{ spectrum of } \Phi_{2\pi}\},$$

where $\Phi_{2\pi}$ is the time 2π map for linear flow. The eigenvalue equation is $v_t = \mathcal{L}(t)v$ for $t \in (0,2\pi)$, and $v(2\pi) = e^{2\pi\sigma}v(0)$. Equivalently, $u(t) = e^{\sigma t}v(t)$, where u solves

$$\sigma u + u_t = u_{xx} - (F_u(\bar{u}(x,t))u)_x, \quad u(x,t) = u(x,t+2\pi) \tag{5}$$

and is spatially localized. We assume the spectrum is as in figure 3 (recall the nonuniqueness of the floquet spectrum), with a double eigenvalue at zero and eigenfunctions $\bar{u}^{\mathrm{per}}_x$ and $\bar{u}^{\mathrm{per}}_t$.

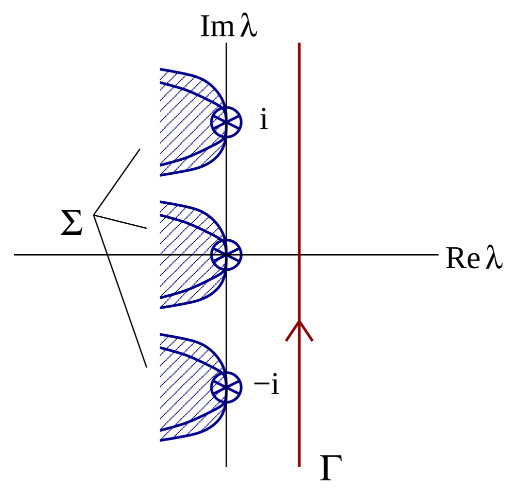


Figure 3: Spectrum Σ for time-periodic shocks and the contour Γ, used in (6).

Take the Laplace transform of (2) and utilize the Fourier expansion

$$F_u(\bar{u}^{\mathrm{per}}(x,t)) = \sum_{k\in\mathbb{Z}} F_k(x)e^{ikt}$$

to obtain

$$\lambda\hat{v}(x,\lambda) - v_0(x) = \partial_x^2\hat{v}(x,\lambda) - \partial_x\left(\sum_{k\in\mathbb{Z}} F_k(x)\hat{v}(x,\lambda - ik)\right).$$

The equations at λ_1 and λ_2 couple only if $\lambda_1 - \lambda_2 \in i\mathbb{Z}$. To exploit this, define σ via

$$\lambda = \sigma + in, \quad -\frac{1}{2} < \mathrm{Im}(\sigma) \le \frac{1}{2}, \quad n \in \mathbb{Z}$$
$$\hat{v}^n(x,\sigma) = \hat{v}(x,\sigma + in),$$

which is motivated by [CCL04] and similar to a Bloch wave decomposition. Thus,

$$(\sigma + in)\hat{v}^n - v_0 = \partial_x^2\hat{v}^n - \partial_x\left(\sum_{k\in\mathbb{Z}} F_k\hat{v}^{n-k}\right)$$

If we could prove the existence of a solution with a well defined Fourier series, $\sum_{n\in\mathbb{Z}} e^{int}\hat{v}^n(x,\sigma)$, then we would have the contour integral representation

$$v(x,t) = \frac{1}{2\pi i}\int_{\mu-i\infty}^{\mu+i\infty} e^{\lambda t}\hat{v}(x,\lambda)d\lambda = \frac{1}{2\pi i}\int_{\mu-i/2}^{\mu+i/2} e^{\sigma t}\sum_{n\in\mathbb{Z}} e^{int}\hat{v}^n(x,\sigma)d\sigma. \tag{6}$$

To show that such a solution exists, we will use exponential dichotomies, which play the role of the resolvent kernel in the stationary case.

Writing (5) as a first order system, we see that the equation corresponding to (3) in the time-periodic case is

$$\partial_x G = \begin{pmatrix} 0 & 1 \\ \partial_t + \sigma + F_{uu}(\bar{u})[\bar{u}_x,\cdot] & F_u(\bar{u}) \end{pmatrix} G + \Delta, \quad \Delta(x-y,t) = \begin{pmatrix} 0 \\ \delta(t)\delta(x-y) \end{pmatrix},$$

which we pose on the space $Y = H^{1+\epsilon}(S^1) \times H^{1/2+\epsilon}(S^1)$.

Using the exponential dichotomy, Φ^{s} and Φ^{u}, of the above equation [SS01] and a Birman-Schwinger type argument, we prove the solution can be written

$$G(x,t,y,\sigma) \sim \int_{-\infty}^x \Phi^{\mathrm{s}}(x,t,z,\sigma)\Delta(z-y,t)dz + \int_{+\infty}^x \Phi^{\mathrm{u}}(x,t,z,\sigma)\Delta(z-y,t)dz.$$

The dichotomy yields solutions that are asymptotic to the stable and unstable subspaces of the asymptotic matrices, which, without the ∂_t, are

$$\begin{pmatrix} 0 & 1 \\ \sigma & F_u(u_\pm) \end{pmatrix}. \tag{7}$$

These are the same for stationary shocks and determine the leading order temporal decay of perturbations. Therefore, the large-time behavior for both types of shocks is essentially the same.

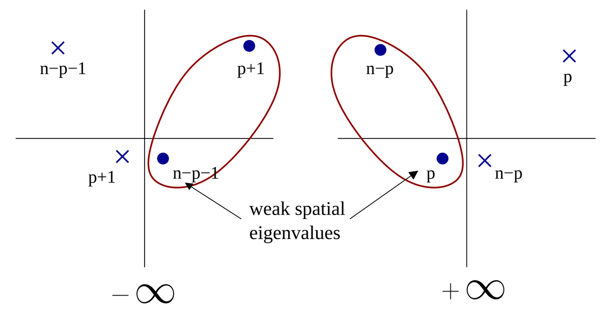


Figure 4: The so-called spatial eigenvalues of (7). Those circled correspond to the decaying directions, picked out by the exponential dichotomy. By (9), the weak eigenvalues, denoted by $\mu(\sigma)$, determine the leading order decay of perturbations.

Using the theory of exponential dichotomies in [SS01], we prove that the following representation, which should be compared with (4), is well defined:

$$\mathcal{G}(x,t,y) = \frac{1}{2\pi i}\int_{\mu-i/2}^{\mu+i/2} e^{\sigma t}G(x,t,y,\sigma)d\sigma = \frac{1}{2\pi i}\int_{\mu-i/2}^{\mu+i/2} e^{\sigma t}\sum_{n\in\mathbb{Z}} e^{int}\hat{G}^n(x,t,y,\sigma)d\sigma. \tag{8}$$

Pointwise estimates for the Green's function

We use the dichotomy to obtain an expansion of $G(x,t,y,\sigma)$, for $\sigma \sim 0$:

$$G(x,t,y,\sigma) \approx -\frac{1}{\sigma}\bar{u}^{\mathrm{per}}_x(x,t)\sum_{a^-_{in}} c_{in,1}e^{-\mu^-_{in}(\sigma)y} - \frac{1}{\sigma}\bar{u}^{\mathrm{per}}_t(x,t)\sum_{a^-_{in}} c_{in,2}e^{-\mu^-_{in}(\sigma)y}$$
$$+\begin{cases} \sum_{a^+_{out},a^-_{in}} c_1 e^{\mu^+_{out}(\sigma)x - \mu^-_{in}(\sigma)y} & \text{for } y \le 0 \le x \\ \sum_{a^-_{out},a^-_{in,out}} c_2 e^{\mu^-_{out}(\sigma)x - \mu^-_{in,out}(\sigma)y} & \text{for } x \le y \le 0 \\ \sum_{a^-_{in,out},a^-_{in}} c_3 e^{\mu^-_{in,out}(\sigma)x - \mu^-_{in}(\sigma)y} & \text{for } y \le x \le 0 \end{cases} \tag{9}$$

The constants $a^\pm_{in,out}$ are the characteristic speeds associated to the incoming and outgoing directions at $x = \pm\infty$, and $\mu^\pm_{in,out}(\sigma)$ denote the corresponding weak spatial eigenvalues in figure 4. The poles at $\sigma = 0$ correspond to the 0 eigenvalues.

Inserting this expansion into (8), we obtain an expansion of the Green's function as $t \to \infty$ of the form $\mathcal{G}(x,t,y) = E_1(x,t,y) + E_2(x,t,y) + \tilde{\mathcal{G}}(x,t,y)$, where

$$E_1(x,t,y) = \bar{u}^{\mathrm{per}}_x(x,t)e_1(y,t), \quad E_2(x,t,y) = \bar{u}^{\mathrm{per}}_t(x,t)e_2(y,t)$$
$$e_i(y,t) \approx \sum_{a^-_{in}} c_{in}\left(\mathrm{errfn}\left(\frac{y + a^-_{in}t}{\sqrt{4t}}\right) - \mathrm{errfn}\left(\frac{y - a^-_{in}t}{\sqrt{4t}}\right)\right), \tag{10}$$

and

$$\tilde{\mathcal{G}}(x,t,y) \approx \sum_{a^-_{out}} \frac{c_1}{\sqrt{4\pi t}}e^{-\frac{(x-y-a^-_{out}t)^2}{4t}} + \sum_{a^-_{out},a^-_{in}} \frac{c_2}{\sqrt{4\pi t}}e^{-\frac{(x-a^-_{out}(t-|y/a^-_{in}|))^2}{4t}}$$
$$+ \sum_{a^-_{in},a^+_{out}} \frac{c_3}{\sqrt{4\pi t}}e^{-\frac{(x-a^+_{out}(t-|y/a^-_{in}|))^2}{4t}}, \tag{11}$$

which is essentially the same as the expansion in [HZ06].

Intuition behind expansion:

- Zero eigenvalues associated to incoming directions; perturbation moves towards shock and determines to which translate (time and space) the solution converges
- As $t \to \infty$, $E_1(x,t,y) \sim \bar{u}^{\mathrm{per}}_x(x) \cdot 1$ (see figure 5); product of the eigenfunction and its adjoint, which is just a constant; like a generalized spectral projection
- Leading order decay determined by $\tilde{\mathcal{G}}$; consists of Gaussians that move along outgoing characteristics (see figure 5); determined by the weak eigenvalues (see figure 4), which are related to the continuous spectrum at the origin.

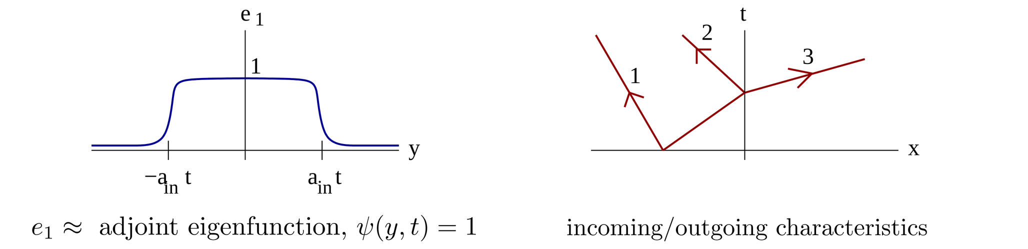


Figure 5: Left: The term e_1 in (10) converges to the constant 1, the adjoint eigenfunction for spatial translation. Right: The 3 pieces of $\tilde{\mathcal{G}}$ in (11) track the motion of the perturbation along the characteristics in the 3 depicted ways.

Nonlinear Analysis

The following Ansatz for equation (1) allows one to exploit the above expansion

$$u(x,t) = \bar{u}^{\mathrm{per}}(x-\rho(t), t-\tau(t)) + v(x-\rho(t),t)$$
$$v_t = \mathcal{L}(t)v + Q(v,v_x)_x + \dot{\rho}(\bar{u}_x + v_x) + \dot{\tau}\bar{u}_t + (\mathcal{O}(\tau)v)_x.$$

Upon writing the integral form of solutions, appropriate choices of $\rho(t)$ and $\tau(t)$ remove the non-decaying terms, $E_{1,2}$. Thus, the evolution of the perturbation v is governed only by the decaying term, $\tilde{\mathcal{G}}$, allowing one to prove nonlinear stability.

Conclusions

The above analysis leads to the following theorem:

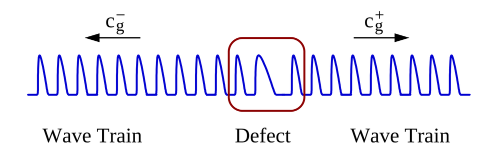
Theorem [BSZ] *Under appropriate spectral stability assumptions, the profile $\bar{u}^{\mathrm{per}}$ is nonlinearly asymptotically stable with respect to initial perturbations u_0 satisfying $\|(1+|\cdot|^2)^{3/4}u_0(\cdot)\|_{H^3} \le \epsilon$ sufficiently small. More precisely, there exist functions $(\rho,\tau)(t)$ and constants (ρ^*,τ^*) such that*

$$||u(\cdot,t) - \bar{u}^{\mathrm{per}}(\cdot - \rho^* - \rho(t), t - \tau^* - \tau(t))||_{L^p} \le C\epsilon(1+t)^{-\frac{1}{2}(1-\frac{1}{p})}, \quad 1 \le p \le \infty$$
$$|(\rho^*,\tau^*)| + (1+t)^{1/2}|(\rho,\tau)(t)| \le C\epsilon,$$

where $u(x,t)$ is the solution to (1) satisfying $u(x,0) = \bar{u}^{\mathrm{per}}(x,0) + u_0(x)$.

Outlook

Nonlinear stability of other types of time-periodic solutions could be analyzed using similar techniques. For example, sources are solutions to reaction-diffusion systems that are spatially asymptotic to two different spatially periodic solutions:



In that case, additional difficulties would result from the spatially periodic, rather than constant, asymptotic matrices in (7). Also, the conservation-law structure of (1) would be absent, and this is crucial for the nonlinear estimates used above.

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