Nonlinear stability of time-periodic viscous shocks

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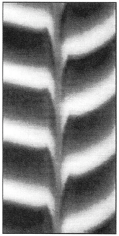
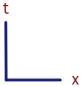
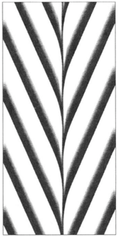
Motivation

Time-periodic patterns in reaction-diffusion systems:







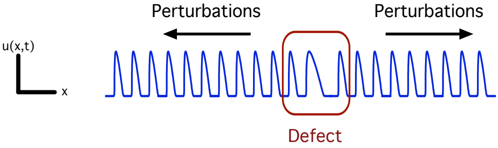
Experiment: chemical reaction
chlorite-iodite-malonic-acid (CIMA)

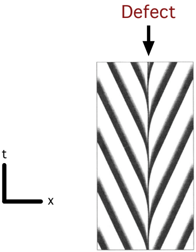
Numerical simulation:
reaction-diffusion equation
$u_t = Du_{xx} + f(u)$

[Perraud et. al., Phys. Rev. Lett. 1993]

Motivation

Diagram of a source:





Importance in applications:

- Widely observed in experiments and numerics
- Defect created spontaneously; not caused by inhomogeneity
- Organize dynamics in rest of spatial domain

Mathematical interest:

- Linear stability: embedded zero eigenvalue; time-periodic linear operator
- Nonlinear stability: weighted spaces don't work, no current methods apply

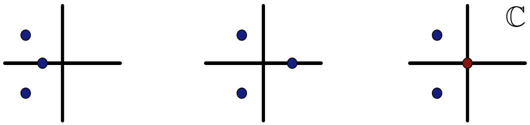
Linear stability overview

$$u_t = \mathcal{F}(u)$$

Stability Ansatz:

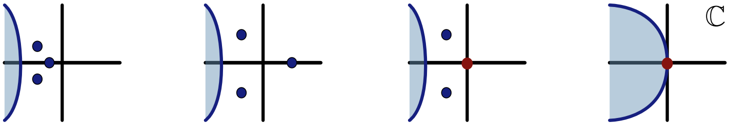
$$u(t) = \bar{u} + v(t) \quad \rightarrow \quad v_t = f_u(\bar{u})v + N(v)$$

ODE: finite dimensions



Stable　　Unstable　　Neutral

PDE: infinite dimensions

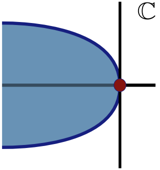


Stable　　Unstable　　Neutral　　Neutral

Stability of sources

Linear stability: two key issues

- Continuous spectrum up to imaginary axis; embedded zero eigenvalue
- Time-periodic operator; no Floquet theory



$$
\begin{aligned}
u_t &= Du_{xx} + f(u) \\
v_t &= \underbrace{Dv_{xx} + f_u(\bar{u}(x,t))v}_{\mathcal{L}(t)v} + N(v)
\end{aligned}
$$

Nonlinear stability:

- Source: perturbations moving outward
- Current methods can't be applied

Focus on linear issues: time-periodic shocks

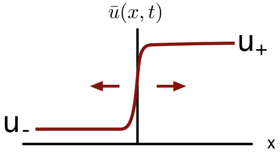
- Same linear issues
- Nonlinearity can be dealt with

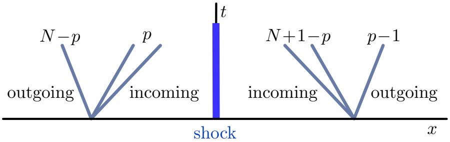
Time-periodic shocks

Parabolic system of conservation laws:

$$u_t = u_{xx} - F(u)_x, \qquad u(t) \in X, \qquad u(x,t) \in \mathbb{R}^N$$

Time-periodic shock: $\bar{u}(x,t)$, $\lim_{x\to\pm\infty} \bar{u}(x,t) = u_\pm$, $\bar{u}(x,t) = \bar{u}(x,t+2\pi)$





Lax shock:

- Eigenvalues $a_1^\pm < \ldots < a_N^\pm$ of $F_u(u_\pm)$ are real, distinct, nonzero.
- There is a number $p \in \{1, \ldots, N\}$ so that

$$a_{N-p}^- < 0 < a_{N-p+1}^-, \qquad a_{N-p+1}^+ < 0 < a_{N-p+2}^+$$

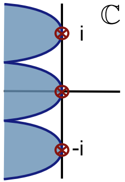
Time-periodic shocks

$$u_t = u_{xx} - F(u)_x$$

Analyze stability: $u(x,t) = \bar{u}(x,t) + v(x,t)$

$$v_t = \mathcal{L}(t)v + N(v)_x, \quad \mathcal{L}(t)v = v_{xx} - (F_u(\bar{u}(x,t))v)_x$$

Assume spectrum of $\mathcal{L}(t)$:



- Double zero eigenvalue; eigenfunctions $\bar{u}_x$ and $\bar{u}_t$
- Motivation: Hopf bifurcation
  [Texier, Zumbrun 08], [Sandstede, Scheel 08]

Stability: solution converges to space and time translate of wave

$$\lim_{t\to\infty} u(x,t) = \bar{u}(x - q^*, t - \tau^*)$$

Time-Periodic Shocks: statement of result

Theorem [B., Sandstede, Zumbrun 08] Under appropriate spectral stability assumptions, the profile $\bar{u}$ is nonlinearly asymptotically stable with respect to perturbations v_0 with $||(1+|\cdot|)^{3/2}v_0(\cdot)||_{H^3} \leq \epsilon$. Also, $\exists\ q(t), \tau(t), q^*, \tau^*$ so that

$$\|u(\cdot,t) - \bar{u}(\cdot - q^* - q(t), t - \tau^* - \tau(t))\|_{L^p(\mathbb{R})} \leq \frac{C\epsilon}{(1+t)^{\frac{1}{2}\left(1-\frac{1}{p}\right)}}$$

for $1 \leq p \leq \infty$, where u is the solution to the conservation law with initial data $u_0(x) = \bar{u}(x,0) + v_0(x)$, and

$$|(q^*, \tau^*)| + (1+t)^{1/2}|(q,\tau)(t)| \leq C\epsilon.$$

Proof:

i) Develop a contour integral representation of linear evolution and derive pointwise bounds

ii) Extract neutral behavior due to zero eigenvalues

iii) Use fixed-point iteration scheme to prove nonlinear stability

Sketch of proof: i) contour integral and pointwise bounds

Recall contour integral for time-independent operators:

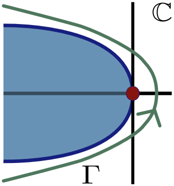
$$v_t = \mathcal{L}v, \qquad v(0) = v_0$$

- Laplace transform: $\hat{v}(x,\lambda) = \int_0^\infty e^{-t\lambda} v(x,t)dt$
- Solve with resolvent

$$\lambda\hat{v} - v_0 = \mathcal{L}\hat{v} \quad \rightarrow \quad \hat{v} = (\lambda - \mathcal{L})^{-1} v_0$$

- Invert Laplace transform

$$v(t) = e^{\mathcal{L}t} v_0 = \frac{1}{2\pi i}\int_\Gamma e^{t\lambda}(\lambda - \mathcal{L})^{-1} v_0 d\lambda$$



Sketch of proof: i) contour integral and pointwise bounds

Pointwise estimates via resolvent kernel, $G(x, y, \lambda)$:

$$\lambda G - \delta(\cdot - y) = \mathcal{L} G$$

so that

$$((\lambda - \mathcal{L})^{-1} v_0)(x) = \int_{\mathbb{R}} G(x, y, \lambda) v_0(y) dy$$

This leads to the Green's function, $\mathcal{G}(x, t, y)$:

$$\mathcal{G}(x, t, y) = \frac{1}{2\pi i} \int_{\Gamma} e^{t\lambda} G(x, y, \lambda) d\lambda$$

$$v(x, t) = \int_{\mathbb{R}} \mathcal{G}(x, t, y) v_0(y) dy$$

Key points:

- Resolvent kernel still defined pointwise inside continuous spectrum
- Can deform contour Γ into spectrum to obtain sharp pointwise bounds
- Stationary shocks: [Zumbrun, Howard 98]

Sketch of proof: i) Floquet spectrum for contour integral

Floquet spectrum: $\Phi_{2\pi}$ is time 2π map for linear flow

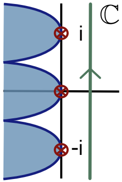
$$\Sigma = \{\sigma \in \mathbb{C} : e^{2\pi\sigma} \in \text{ spectrum of } \Phi_{2\pi}\}$$

Eigenvalue equation: $v_t = \mathcal{L}(t)v$ for $t \in (0, 2\pi)$, $v(2\pi) = e^{2\pi\sigma} v(0)$

Equivalently, $v(t) = e^{\sigma t} u(t)$, u is spatially localized and solves

$$\sigma u + u_t = u_{xx} - (F_u(\bar{u}(x,t))u)_x, \quad u(x,t) = u(x, t+2\pi)$$

Two eigenfunctions for $\sigma = 0$: $\bar{u}_x(x,t)$ and $\bar{u}_t(x,t)$



Sketch of proof: i) definition of "resolvent kernel"

Recall: Green's function solved inhomogeneous eigenvalue equation:

$$(\lambda - \mathcal{L})\hat{v} = v_0, \qquad \hat{v}(x) = \int_{\mathbb{R}} G(x,y,\lambda) v_0(y) dy$$

Floquet eigenvalue equation:

$$\sigma v + v_t = v_{xx} - (F_u(\bar{u})v)_x, \quad v(x,t) = v(x,t+2\pi)$$

Write as first-order spatial dynamical system:

$$V_x = \begin{pmatrix} 0 & 1 \\ \partial_t + \sigma + F_{uu}(\bar{u})[\bar{u}_x, \cdot] & F_u(\bar{u}) \end{pmatrix} V, \qquad V = \begin{pmatrix} v \\ v_x \end{pmatrix}$$

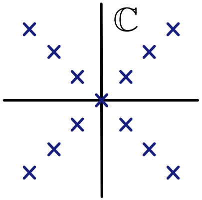
Work in $Y = H^1(S^1) \times H^{1/2}(S^1)$.

Inhomogeneous equation:

$$V_x = \begin{pmatrix} 0 & 1 \\ \partial_t + \sigma + F_{uu}(\bar{u})[\bar{u}_x, \cdot] & F_u(\bar{u}) \end{pmatrix} V + \mathcal{F}$$

Sketch of proof: i) exponential dichotomies

Ill-posed equation: $U_x = \begin{pmatrix} 0 & 1 \\ \partial_t & 0 \end{pmatrix} U$ has eigenvalues $\pm\sqrt{ik}$, $k \in \mathbb{Z}$



Define exponential dichotomies on Y [Sandstede and Scheel 01]:

$$\partial_x \Phi^{s,u} = \mathcal{A}(x, \partial_t, \sigma)\Phi^{s,u}, \quad \Phi^{s,u}(x, y, \sigma) \in L(Y, Y)$$

$$|\Phi^s(x, y, \sigma)V|_Y \leq Ke^{-\eta(x-y)}|V|_Y, \text{ for } x \geq y$$

$$|\Phi^u(x, y, \sigma)V|_Y \leq Ke^{-\eta(y-x)}|V|_Y, \text{ for } y \geq x$$

Solution operators in forwards and backwards "time"

Sketch of proof: i) definition of "resolvent kernel"

"Resolvent kernel" defines solutions, for arbitrary $\mathcal{F}$, to

$$V_x = \begin{pmatrix} 0 & 1 \\ \partial_t + \sigma + F_{uu}(\bar{u})[\bar{u}_x, \cdot] & F_u(\bar{u}) \end{pmatrix} V + \mathcal{F}$$

Solve using exponential dichotomies [Sandstede, Scheel 01]:

$$V(x,\sigma) = \int_{-\infty}^{x} \Phi^s(x,y,\sigma)\mathcal{F}(y)dy + \int_{+\infty}^{x} \Phi^u(x,y,\sigma)\mathcal{F}(y)dy$$

Work in $Y = H^1(S^1) \times H^{1/2}(S^1)$, so well defined Fourier series:

$$V(x,\sigma,t) = \sum_n e^{int}\hat{V}^n(x,\sigma)$$

This will give convergence of the contour integral.

Sketch of proof: i) contour integral

Recall time-independent case:

$$\lambda G - \delta(\cdot - y) = \mathcal{L}G, \quad \mathcal{G}(x,t,y) = \frac{1}{2\pi i}\int_\Gamma e^{t\lambda} G(x,y,\lambda)d\lambda$$

Need to solve:

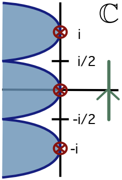
$$\partial_x G = \begin{pmatrix} 0 & 1 \\ \partial_t + \sigma + F_{uu}(\bar{u})[\bar{u}_x, \cdot] & F_u(\bar{u}) \end{pmatrix} G + \Delta \qquad (*)$$

for

$$\Delta(x,t) = \begin{pmatrix} 0 \\ \delta(t)\delta(x-y) \end{pmatrix}$$

Green's function:

$$\begin{aligned}\mathcal{G}(x,t,y) &= \frac{1}{2\pi i}\int_{\mu - i\infty}^{\mu + i\infty} e^{\lambda t} G(x,y,\lambda)d\lambda \\ &= \frac{1}{2\pi i}\int_{\mu - \frac{i}{2}}^{\mu + \frac{i}{2}} e^{\sigma t} \underbrace{\sum_{n\in\mathbb{Z}} e^{int}\hat{G}^n(x,y,\sigma)}_{G(x,y,\sigma,t) \text{ solves } (*)} d\sigma\end{aligned}$$



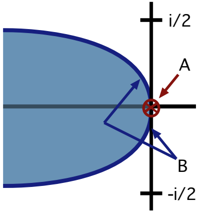
Sketch of proof: i) pointwise bounds

Green's function:

$$\mathcal{G}(x,t,y) = \frac{1}{2\pi i}\int_{\mu-\frac{i}{2}}^{\mu+\frac{i}{2}} e^{\sigma t} G(x,y,\sigma,t)\,d\sigma$$

Meromorphically extend "resolvent kernel" near $\sigma \sim 0$:

$$G(x,y,\sigma,t) \sim A + B$$



- A:
  - Zero eigenvalues
  - Similar to a spectral projection
- B:
  - Continuous spectrum
  - Related to weak spatial decay of G

Sketch of proof: i) B, continuous spectrum

Recall G solves:

$$\partial_x G = \begin{pmatrix} 0 & 1 \\ \partial_t + \sigma + F_{uu}(\bar{u})[\bar{u}_x, \cdot] & F_u(\bar{u}) \end{pmatrix} G + \Delta$$

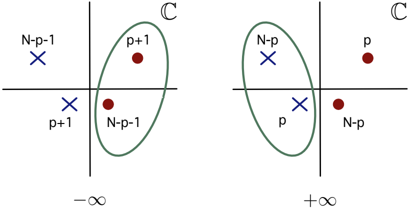
Asymptotic limits of operator:

$$\begin{pmatrix} 0 & 1 \\ \partial_t + \sigma & F_u(u_\pm) \end{pmatrix} \quad \Rightarrow \quad \begin{pmatrix} 0 & 1 \\ \sigma & F_u(u_\pm) \end{pmatrix}$$

Spatial eigenvalues for $\sigma \approx 0$:

$$\nu \sim a_j^\pm$$

$$\nu \sim -\frac{\sigma}{a_j^\pm}$$

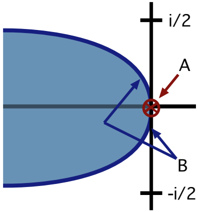


- Weak decay becomes weak growth as σ crosses into essential spectrum
- Leads to leading order decay of Green's function

Sketch of proof: i) A, eigenvalues

Meromorphically extend "resolvent kernel" near $\sigma \sim 0$:

$$G(x, y, \sigma, t) \sim A + B$$



A: eigenvalues, similar to a spectral projection

$$\begin{aligned} A &\approx \frac{1}{\sigma}\bar{u}_x(x,t)\langle\psi_1(y),\cdot\rangle + \frac{1}{\sigma}\bar{u}_t(x,t)\langle\psi_2(y),\cdot\rangle \\ &\approx \frac{1}{\sigma}\bar{u}_x(x,t)\sum_{a^-_{\mathrm{in}}} c_1(y)e^{(\sigma/a^-_{\mathrm{in}})y} + \frac{1}{\sigma}\bar{u}_t(x,t)\sum_{a^-_{\mathrm{in}}} c_2(y)e^{(\sigma/a^-_{\mathrm{in}})y} \end{aligned}$$

Sketch of proof: i) pointwise bounds

Invert Laplace transform:

$$
\begin{aligned}
\mathcal{G}(x,t,y) &= \frac{1}{2\pi i}\int_{\mu-\frac{i}{2}}^{\mu+\frac{i}{2}} e^{\sigma t} G(x,y,\sigma,t)\,d\sigma \\
&= \frac{1}{2\pi i}\int_{\mu-\frac{i}{2}}^{\mu+\frac{i}{2}} e^{\sigma t}(A+B)\,d\sigma \\
&= \underbrace{[\mathcal{E}_1(x,t,y)+\mathcal{E}_2(x,t,y)]}_{A} + \underbrace{\tilde{\mathcal{G}}(x,t,y)}_{B}
\end{aligned}
$$

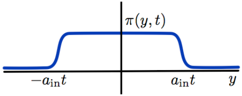
Sketch of proof: i) pointwise bounds

Invert Laplace transform:

$$\begin{aligned}
\mathcal{G}(x,t,y) &= \frac{1}{2\pi i}\int_{\mu-\frac{i}{2}}^{\mu+\frac{i}{2}} e^{\sigma t} G(x,y,\sigma,t)\,d\sigma \\
&= \frac{1}{2\pi i}\int_{\mu-\frac{i}{2}}^{\mu+\frac{i}{2}} e^{\sigma t} (A+B)\,d\sigma \\
&= [\mathcal{E}_1(x,t,y)+\mathcal{E}_2(x,t,y)] + \tilde{\mathcal{G}}(x,t,y)
\end{aligned}$$

Eigenfunction terms:

$$\begin{aligned}
\mathcal{E}_i(x,t,y) &= \bar{u}_i(x,t)\pi_i(y,t) \\
\pi_i(y,t) &= \sum_{a_{in}} c_i \left[\mathrm{errfn}\left(\frac{y+a_{in}t}{\sqrt{4(t+1)}}\right) - \mathrm{errfn}\left(\frac{y-a_{in}t}{\sqrt{4(t+1)}}\right)\right]
\end{aligned}$$

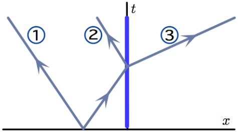


Sketch of proof: i) pointwise bounds

Invert Laplace transform:

$$\begin{aligned}
\mathcal{G}(x,t,y) &= \frac{1}{2\pi i}\int_{\mu-\frac{i}{2}}^{\mu+\frac{i}{2}} e^{\sigma t} G(x,y,\sigma,t)d\sigma \\
&= \frac{1}{2\pi i}\int_{\mu-\frac{i}{2}}^{\mu+\frac{i}{2}} e^{\sigma t}(A+B)d\sigma \\
&= [\mathcal{E}_1(x,t,y)+\mathcal{E}_2(x,t,y)] + \tilde{\mathcal{G}}(x,t,y)
\end{aligned}$$

$$\begin{aligned}
\tilde{\mathcal{G}}(x,t,y) \approx & \sum_{a^-_{out}} \frac{c_1}{\sqrt{4\pi t}} e^{-\frac{(x-y-a^-_{out}t)^2}{4t}} + \sum_{a^-_{out},a^-_{in}} \frac{c_2}{\sqrt{4\pi t}} e^{-\frac{(x-a^-_{out}(t-|y/a^-_{in}|))^2}{4t}} \\
& + \sum_{a^-_{in},a^+_{out}} \frac{c_3}{\sqrt{4\pi t}} e^{-\frac{(x-a^+_{out}(t-|y/a^-_{in}|))^2}{4t}}
\end{aligned}$$



Sketch of proof: ii) neutral behavior, iii) nonlinear stability

Remove neutral part: define time-dependent phase shift for space and time

$$u(x,t) = \bar{u}(x - q(t), t - \tau(t)) + v(x,t)$$
$$v_t = \mathcal{L}(t)v + N(v)_x + \dot{q}(\bar{u}_x + v_x) + \dot{\tau}\bar{u}_t + (\mathcal{O}(\tau)v)_x$$

Using variation of constants, solution must satisfy: $\mathcal{G} = \bar{u}_x\pi_1 + \bar{u}_t\pi_2 + \tilde{\mathcal{G}}$

$$\begin{aligned}
q(t) &= -\int_{\mathbb{R}} \pi_1 v_0 dy + \int_0^t \int_{\mathbb{R}} (\pi_1)_y (N + \dot{q}v) dyds + \int_0^t \int_{\mathbb{R}} (\pi_1)_y (\mathcal{O}(\tau)v) dyds \\
\tau(t) &= -\int_{\mathbb{R}} \pi_2 v_0 dy + \int_0^t \int_{\mathbb{R}} (\pi_2)_y (N + \dot{q}v) dyds + \int_0^t \int_{\mathbb{R}} (\pi_2)_y (\mathcal{O}(\tau)v) dyds \\
v(x,t) &= \int_{\mathbb{R}} \tilde{\mathcal{G}} v_0 dy - \int_0^t \int_{\mathbb{R}} \tilde{\mathcal{G}}_y (N + \dot{q}v) dyds - \int_0^t \int_{\mathbb{R}} \tilde{\mathcal{G}}_y (\mathcal{O}(\tau)v) dyds
\end{aligned}$$

Set up fixed point iteration scheme to prove existence and convergence result.

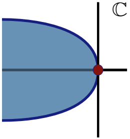
Summary and Extensions

Goal: prove nonlinear stability of time-periodic shocks

$$u(x,t) = \bar{u}(x,t) + v(x,t), \quad \lim_{t\to\infty} u(x,t) = \bar{u}(x-q^*, t-\tau^*)$$

Mathematical issues overcome

- Continuous spectrum up to imaginary axis; embedded zero eigenvalue
- Time-periodic operator; no Floquet theory



$$v_t = \mathcal{L}(t)v + N(v)_x$$

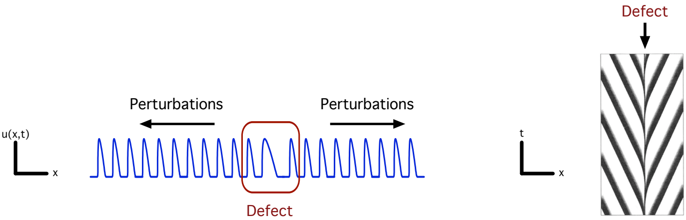
Extensions:

- Non-Lax shocks
- Real viscosity, $u_{xx} \to (B(u)u_x)_x$
- Sources...

Outlook: sources in reaction-diffusion systems

$$u_t = Du_{xx} + f(u)$$

Source:



Mathematical Issues:

- Similar spectral and linear stability issues
- Spatial end states are spatially-periodic: spatial Floquet spectrum
- No conservation-law structure, $F(u)_x$; used in nonlinear estimates