MA 122
Summer II 2017
Final Exam Keys
08/10/2017
Time Limit: 150 Minutes

Name (Print): ____________________

This exam contains 10 pages (including this cover page) and 8 problems. Check to see if any pages are missing. Enter all requested information on the top of this page.

You can **NOT** use your books or any calculators on this exam, but you can take one page of notes.

You are required to show your work on each problem on this exam. The following rules apply:

- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- The last problem is for **extra credits**. It is **much more difficult**, so only attempt it if you have completed the other problems as best you can.
- If you need more space, use the back of the pages; clearly indicate when you have done this.

Do not write in the table to the right.

Problem	Points	Score
1	10	
2	10	
3	15	
4	10	
5	15	
6	20	
7	20	
8	10	
Total:	110	

1. (10 points) Find $f^{(n)}(x)$, where $f(x) = \frac{1}{2x+1}$.

$f^{(n)}(x) = n!(-2)^n(1+2x)^{-(n+1)}$

2. (10 points) If y is defined implicitly by $x^2 + y^2 = C$, show that y is the general solution to the differential equation $yy' = -x$. Then, find and graph the particular solutions satisfying $y(0) = 0$, $y(0) = 1$, and $y(0) = 2$. Can you describe what the family of general solution looks like geometrically?

Geometrically, the general solution is a family of circles with center $(0, 0)$.

3. (15 points) There are 3 points $(-1,-1)$, $(1,1)$ and $(2,8)$, obtained from $f(x)=x^3$.

   (a) (5 points) Find the interpolating polynomial for these data using Divided Difference Table.

   $p(x) = -1 + (x+1) + 2(x+1)(x-1) = 2x^2 + x - 2$

   (b) (5 points) Find the expression of piecewise linear approximation for these data.

$$L(x) = \begin{cases} x & -1 \le x < 1 \\ 7x - 6 & 1 \le x \le 2 \end{cases}$$

   (c) (5 points) Graph the functions obtained from (a), (b) and $f(x)=x^3$ in one figure in the domain $-1 \le x \le 2$. Use the results from (a) and (b) to approximate the function value at $x = 0$. In this case, which approximation method is better? Explain why.

   Since $p(0) = -2$, $L(0) = 0$ and $f(0) = 0$, in this case, linear approximation is better.

4. (10 points) Find the 3rd-degree Taylor polynomial at $\mathbf{\frac{1}{3}}$ for $f(x) = \ln(3x)$ and use it to approximate $f(1)$. Is it a good approximation? ($\ln 3 \approx 1.0986$)

$p_3(x) = 3(x - \frac{1}{3}) - \frac{9}{2}(x - \frac{1}{3})^2 + 9(x - \frac{1}{3})^3$

Since $p_3(1) = \frac{8}{3} = 2.6667$ and $f(1) \approx 1.0986$, it is a really bad approximation, because the interval of convergence is $0 < x < \frac{2}{3}$.

5. (15 points) (a) (5 points) Find the Taylor series at 0 for e^{-2x^2} and interval of convergence.

$e^{-2x^2} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}(2x^2)^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} 2^n x^{2n}$

Interval of convergence is $-\infty < x < \infty$.

(b) (5 points) Find the Taylor series at 0 for $\int e^{-2x^2}\ dx$ and interval of convergence.

$\int e^{-2x^2}\ dx = \sum_{n=0}^{\infty} \int \frac{(-1)^n}{n!} 2^n x^{2n}\ dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} 2^n \frac{1}{2n+1} x^{2n+1} + C$

Interval of convergence is $-\infty < x < \infty$.

(c) (5 points) Approximate the integral $\int_0^1 e^{-2x^2}\ dx$ to within ± 0.05.($\frac{2}{3} \approx 0.6667$, $\frac{4}{21} \approx 0.1905$ and $\frac{2}{27} \approx 0.0741$)

$\int_0^1 e^{-2x^2}\ dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} 2^n \frac{1}{2n+1}$

$|\int_0^1 e^{-2x^2}\ dx - 1| < |-2/3| > 0.0005$

$|\int_0^1 e^{-2x^2}\ dx - (1 - 2/3)| < |2^2/5/2!| > 0.0005$

$|\int_0^1 e^{-2x^2}\ dx - (1 - 2/3 + 2^2/5/2!)| < |-2^3/7/3!| > 0.0005$

$|\int_0^1 e^{-2x^2}\ dx - (1 - 2/3 + 2^2/5/2! - 2^3/7/3!)| < |2^4/9/4!| > 0.0005$

$|\int_0^1 e^{-2x^2}\ dx - (1 - 2/3 + 2^2/5/2! - 2^3/7/3! + 2^4/9/4!)| < |2^5/11/5!| < 0.0005$

Thus, $\int e^{-2x^2}\ dx \approx 1 - 2/3 + 2^2/5/2! - 2^3/7/3! + 2^4/9/4! \approx 0.4497$

6. (20 points) Given a differential equation $y' = 2x(y+1)$.

   (a) (10 points) Find the general solution using the method of "Separation of Variables".

   $y = Ce^{x^2} - 1$

   (b) (10 points) Find the general solution using the method for solving "First-Order Linear Differential Equations".

   $y = Ce^{x^2} - 1$

7. (20 points) The number of words per minute, y, a person can type increases with practice. Suppose the rate of change of N is proportional to the difference between y and an upper limit of 200. Assume that a beginner cannot type at all and a person can type 60 words per minute after 10 hours of practice.

   (a) (5 points) Translate the problem in the language of differential equation with initial condition. What is the model you are using? (In this part, you don't need to determine the rate.)

   Suppose $y(t)$ is the number of words per minute a person can type after t hours, and k is the learning rate. Then, we can rewrite the problem in the following way, and this is the Limited Growth Model.

$$\begin{cases} y'(t) & = k(200 - y) \\ y(0) & = 0 \\ y(10) & = 60 \end{cases}$$

   (b) (10 points) Solve the differential equation in (a) with initial condition. Determine the learning rate. ($\ln 0.7 \approx -0.3567$ and $\ln 0.3 \approx -1.2040$)

   $y = 200(1 - e^{-kt})$, where $k = 0.03567$.

(c) (5 points) What is the equilibrium of the system? Is it dynamically stable? Explain why.

Equilibrium is $y = 200$ and since $\lim_{t\to\infty} y(t) = 200$, it is dynamically stable.

8. (10 points) (*Extra Credits*) Draw the slope field of $y' = x^2 + y^2$. What can you say about the solution curve? Based on the slope field, draw the particular solution satisfying $y(1) = 0$.

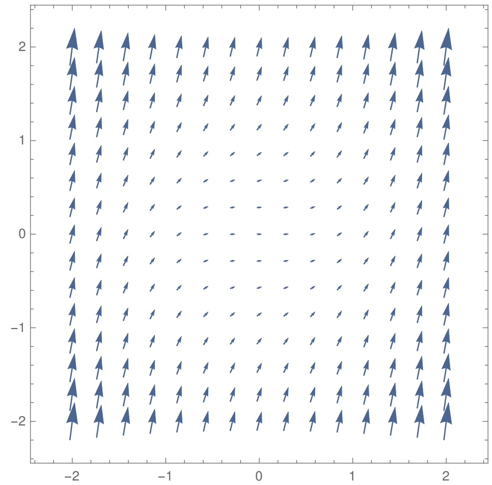

Figure 1: Slope Field

The solution curve is always strictly increasing except at $(0, 0)$.