MA 122
Summer II 2017
Practice Final Exam Keys

Name (Print): ______________________

This exam contains 10 pages (including this cover page) and 8 problems. Check to see if any pages are missing. Enter all requested information on the top of this page.

You can **NOT** use your books or any calculators on this exam, but you can take one page of notes.

You are required to show your work on each problem on this exam. The following rules apply:

- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- The last problem is for **extra credits**. It is **much more difficult**, so only attempt it if you have completed the other problems as best you can.
- If you need more space, use the back of the pages; clearly indicate when you have done this.

Do not write in the table to the right.

Problem	Points	Score
1	10	
2	10	
3	15	
4	10	
5	15	
6	15	
7	25	
8	10	
Total:	110	

1. (10 points) Find $f^{(n)}(x)$, where $f(x) = e^{2x}$. [Suggested Practice: Chap 2.1 - 31, 34, 36.]

$f^{(n)}(x) = 2^n e^{2x}$

2. (10 points) If y is defined implicitly by $y + e^{y^2} - x^2 = C$, show that y is the general solution to the differential equation $(1+2ye^{y^2})y' = 2x$. Then, find a particular solution satisfying $y(1) = 0$. [Suggested Practice: Chap 1.1 - 31, 34, 36.]

Differentiate the general solution implicitly, and you'll get

$$(1 + 2ye^{y^2})y' = 2x$$

Particular solution:

$$y + e^{y^2} - x^2 = 0$$

3. (15 points) There are 3 points $(1, 1)$, $(2, \frac{1}{2})$ and $(4, \frac{1}{4})$, obtained from $f(x) = \frac{1}{x}$. [Suggested Practice: Chap D.1 - 13, 37.]

   (a) (5 points) Find the interpolating polynomial for these data using Divided Difference Table.

   By using Divided Difference Table, you'll get the Newton's form $p(x) = 1 - \frac{1}{2}(x-1) + \frac{1}{8}(x-1)(x-2) = \frac{1}{8}x^2 - \frac{7}{8}x + \frac{7}{4}$.

   (b) (5 points) Find the expression of piecewise linear approximation for these data.

$$L(x) = \begin{cases} -\frac{1}{2}x + \frac{3}{2} & 1 \le x < 2 \\ -\frac{1}{8}x + \frac{3}{4} & 2 \le x \le 4 \end{cases}$$

   (c) (5 points) Graph the functions obtained from (a), (b) and $f(x) = \frac{1}{x}$ in one figure in the domain $1 \le x \le 4$. Use the results from (a) and (b) to approximate the function value at $x = 3$. Which approximation method is better? Explain why.

   Graph here is omitted.

   Since $f(3) = \frac{1}{3}$, $L(3) = \frac{3}{8}$ and $p(3) = \frac{1}{4}$, it turns out that interpolating polynomial approximation is better for our example because the error is smaller.

4. (10 points) Given $(\sin x)' = \cos x$ and $(\cos x)' = -\sin x$, find the 3rd-degree Taylor polynomial at 0 for $f(x) = 2\sin x \cos x$ and use it to approximate $f(\frac{\pi}{6})$. Is it a good approximation? [Suggested Practice: Chap 2.1 - 13, 14.]

First, find the derivatives,

$f(x) = 2\sin x \cos x$, $f'(x) = 2(\cos^2 x - \sin^2 x)$, $f''(x) = -8\sin x \cos x$ and $f'''(x) = -8(\cos^2 x - \sin^2 x)$.

Then, we have, $f(0) = 0$, $f'(0) = 2$, $f''(0) = 0$ and $f'''(0) = -8$.

Thus, the 3rd-degree Taylor polynomial at 0 is $p_3(x) = 2x - \frac{4}{3}x^3$.

Since $f(\frac{\pi}{6}) = \frac{\sqrt{3}}{2} \approx 0.8660$ and $p_3(\frac{\pi}{6}) \approx 0.8558$, it is a good approximation with error less than 0.05.

5. (15 points) [Suggested Practice: Chap 2.3 - 33, 40, 49; Chap 2.4 - 26, 28.]

(a) (5 points) Find the Taylor series at 0 for $\frac{1}{9+x^2}$ and interval of convergence.

$$\frac{1}{9+x^2} = \frac{1}{9}\frac{1}{1+\frac{x^2}{9}} = \frac{1}{9}\sum_{n=0}^{\infty}(-1)^n\left(\frac{x^2}{9}\right)^n = \sum_{n=0}^{\infty}(-1)^n\frac{1}{9^{n+1}}x^{2n}$$

Interval of convergence is $-1 < \frac{x^2}{9} < 1$, which is equivalent to $-3 < x < 3$.

(b) (5 points) Find the Taylor series at 0 for $\int \frac{1}{9+x^2}\,dx$ and interval of convergence.

$$\int \frac{1}{9+x^2}\,dx = \sum_{n=0}^{\infty}\int(-1)^n\frac{1}{9^{n+1}}x^{2n}\,dx = \sum_{n=0}^{\infty}(-1)^n\frac{1}{9^{n+1}}\frac{1}{2n+1}x^{2n+1} + C$$

Interval of convergence is $-3 < x < 3$.

(c) (5 points) Approximate the integral $\int_0^1 \frac{1}{9+x^2}\,dx$ to within ± 0.0005.

$$\int_0^1 \frac{1}{9+x^2}\,dx = \sum_{n=0}^{\infty}(-1)^n\frac{1}{9^{n+1}}\frac{1}{2n+1}$$

$$|R_1(1)| < \frac{1}{9^2}\frac{1}{3} > 0.0005$$

$$|R_3(1)| < \frac{1}{9^3}\frac{1}{5} < 0.0005$$

Thus, $\int_0^1 \frac{1}{9+x^2}\,dx \approx \frac{1}{9^1}\frac{1}{1} + \frac{1}{9^2}\frac{1}{3} \approx 0.1152$

6. (15 points) Given a differential equation $xy' + y = 2x$. [Suggested Practice: Chap 1.3 - 14, 15.]

   (a) (10 points) Find the general solution.

   $y = x + \frac{C}{x}$

   (b) (5 points) Find a particular solution satisfying $y(1) = 1$.

   $y = x$

7. (25 points) A rumor spreads through a population of 1000 people at a rate proportional to the product of the number who have heard it and the number who have not heard it. Suppose 5 people initiated a rumor and 10 people had heard it after 1 day. [Suggested Practice: Chap 1.2 - 40, 45, 68, 69.]

   (a) (5 points) Translate the problem in the language of differential equation with initial condition? What is the model you are using? (In this part, you don't need to determine the rate.)

   Suppose $y(t)$ is the number of people who have heard the rumor after t days, and k is the spread rate. Then, we can rewrite the problem in the following way, and this is the Logistic Growth Model.

$$\begin{cases} y'(t) & = ky(1000 - y) \\ y(0) & = 5 \\ y(1) & = 10 \end{cases}$$

   (b) (10 points) Solve the differential equation in (a) with initial condition. Determine the spread rate based on the information in the problem.

   Using the method of Separation of Variables and plugging in the initial conditions, we'll get

   $y(t) = \frac{1000}{199(\frac{99}{199})^t + 1}$

(c) (5 points) What is the equilibrium of the system? Is it dynamically stable? Explain why.

Equilibrium : $y = 0$ and $y = 1000$.

Since $\lim_{t\to\infty} y(t) = 1000$, $y = 0$ is not dynamically stable but $y = 1000$ is dynamically stable.

(d) (5 points) How many people will have heard the rumor after 7 days? How long will it take for 850 people to hear the rumor? (You don't need to calculate the final number, but you need to provide the final expression of the answer.)

The first answer is $y(7)$ and the second answer is to solve $y(t) = 850$ for t.

8. (10 points) (*Extra Credits*) Draw the slope field of $y' = (x-1)(y-1)$. Based on the slope field, draw the particular solution satisfying $y(2) = 2$.

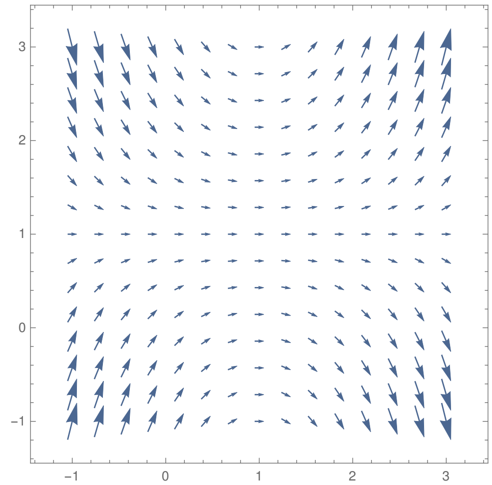

Figure 1: Slope Field