MA 122
Summer II 2017
Midterm Exam Keys
07/20/2017
Time Limit: 150 Minutes

Name (Print): ____________________

This exam contains 14 pages (including this cover page) and 8 problems. Check to see if any pages are missing. Enter all requested information on the top of this page.

You can **NOT** use your books or any calculators on this exam, but you can take one page of notes.

You are required to show your work on each problem on this exam. The following rules apply:

- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- The last problem is for **extra credits**. It is **much more difficult**, so only attempt it if you have completed the other problems as best you can.
- If you need more space, use the back of the pages; clearly indicate when you have done this.

Do not write in the table to the right.

Problem	Points	Score
1	10	
2	10	
3	10	
4	5	
5	15	
6	25	
7	25	
8	10	
Total:	110	

1. (10 points) Find $f_y(0,1)$ and $f_x(1,1)$, where $f(x,y) = e^{2x+y}\ln y^3$.

Answer:

Note that $f(x,y) = 3e^{2x+y}\ln y$.

$f_y(x,y) = 3e^{2x+y}\ln y + 3e^{2x+y}/y$, so $f_y(0,1) = 3e$.

$f_x(x,y) = 6e^{2x+y}\ln y$, so $f_y(1,1) = 0$.

2. (10 points) Find $\int f(x,y)\ \mathrm{d}x$ and $\int f(x,y)\ \mathrm{d}y$, where $f(x,y) = e^{\cos x}\sin x \sin 2y$.

Answer:

choose $u = \cos x$, then $\mathrm{d}u = -\sin x \mathrm{d}x$.

$\int f(x,y)\ \mathrm{d}x = \sin 2y \int -e^u\ \mathrm{d}u = -e^u \sin 2y + C(y) = -e^{\cos x}\sin 2y + C(y)$.

$\int f(x,y)\ \mathrm{d}y = e^{\cos x}\sin x \int \sin 2y\ \mathrm{d}y = -\frac{1}{2}e^{\cos x}\sin x \cos 2y + C(x)$.

3. (10 points) Method of Least Squares.

   (a) (8 points) Find the least squares line referring to $n = 3$ data points $(x_1, y_1) = (0, 1)$, $(x_2, y_2) = (1, 1)$, and $(x_3, y_3) = (2, 0)$.

   Answer:

   Use the least squares formula, you will get $a = -\frac{1}{2}$, $b = \frac{7}{6}$. The line of equation is $y = -\frac{1}{2}x + \frac{7}{6}$.

   (b) (2 points) Use the line in part (a) to estimate y when $x = 10$.

   By plugging $x = 5$ into $y = -\frac{1}{2}x + \frac{7}{6}$, we will have $y = -\frac{23}{6}$.

4. (5 points) Find the solution set of each equation on the given interval.

(a) (2 points) $\cos x \geq 0$ on $[0, 2\pi]$.

Answer:

$[0, \pi/2] \cup [3\pi/2, 2\pi]$.

(b) (3 points) $|\cos 2x| \geq \frac{1}{2}$ on $[0, \pi]$.

Answer:

$[0, \pi/6] \cup [\pi/3, 2\pi/3] \cup [5\pi/6, \pi]$

5. (15 points) Consider the function $f(x) = \cot x$.

(a) (4 points) Find the domain, range and period of $f(x)$.

Answer:

Domain is $x \neq n\pi$, where n is any integer; Range is $(-\infty, +\infty)$; Period is π.

(b) (4 points) Find $f'(x)$.

Answer:

$f'(x) = -\frac{1}{\sin^2 x}$.

(c) (3 points) Graph $y = f(x)$ on the interval $(0, \pi)$.

Answer:

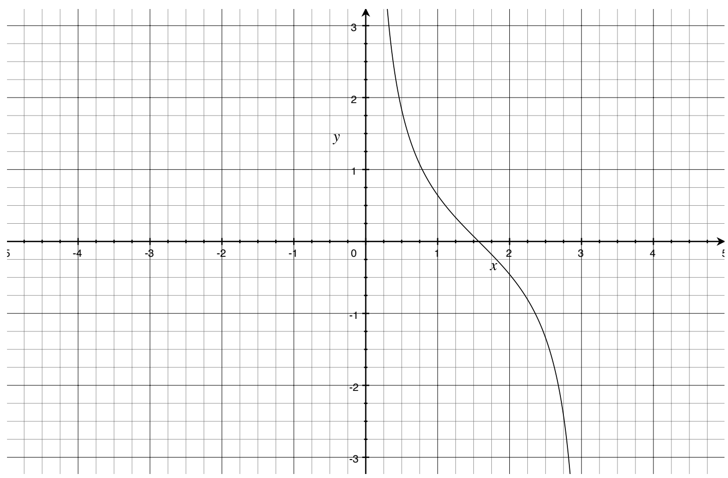


Figure 1: $y = \cot x,\ x \in (0, \pi)$

(d) (4 points) Find $\int f(x)\ \mathrm{d}x$.

Answer:

Choose $u = \sin x$, then $\mathrm{d}u = \cos x \mathrm{d}x$

$\int f(x)\ \mathrm{d}x = \int \frac{1}{u} \mathrm{d}u = \ln |u| + C = \ln |\sin x| + C.$

6. (25 points) Consider the function $f(x, y) = x^2 - x^2y + 2y^2$.

   (a) (5 points) Find the equations of the cross sections of the surface $z = f(x, y)$ produced by cutting it with planes $x = 0$ and $y = 0$ and graph the curves. Can you guess whether point (0, 0) is a local minima point, local maxima point or a saddle point based on these two cross sections? Explain why.

   Answer:

   $z = 2y^2$ and $z = x^2$, (Graphs here are omitted.)

   Local Minima Point, because it is a minima point in each plane.

(b) (10 points) Use the Second-Derivative Test for Local Extrema to find local extrema of $z = f(x, y)$. Does it coincide with the result you got from part (a)?

Answer:

Since $f_x(x, y) = 2x - 2xy$ and $f_y(x, y) = -x^2 + 4y$, we have critical points (0, 0) and $(\pm 2, 1)$.

For (0,0),

Since $f_{xx}(0, 0) = 2$, $f_{xy}(0, 0) = 0$ and $f_{yy}(0, 0) = 4$, we can conclude that (0, 0) is a local minima point with minimal value $f(0, 0) = 0$ based on Second-Derivative Test for Local Extrema, as expected in part (a).

For (2,1),

Since $f_{xx}(2, 1) = 0$, $f_{xy}(2, 1) = -4$ and $f_{yy}(2, 1) = 4$, we can conclude that (2, 1) is a saddle point based on Second-Derivative Test for Local Extrema.

For (-2,1),

Since $f_{xx}(-2, 1) = 0$, $f_{xy}(-2, 1) = -4$ and $f_{yy}(-2, 1) = 4$, we can conclude that (-2, 1) is a saddle point based on Second-Derivative Test for Local Extrema.

(c) (10 points) Minimize $f(x,y)$, subject to $x^2+y^2=5$.

Answer:

$F(x,y,\lambda)=x^2-x^2y+2y^2+\lambda(x^2+y^2-5)$.

Since $F_x(x,y,\lambda)=2x-2xy+2x\lambda$, $F_y(x,y,\lambda)=-x^2+4y+2y\lambda$ and $F_\lambda(x,y,\lambda)=x^2+y^2-5$, we have critical points $(0,\pm\sqrt{5})$, $(\pm 2,1)$ and $(\pm\frac{2\sqrt{5}}{3},-\frac{5}{3})$.

$f(x,y)$ on these critical points are equal to10, 2 and $\frac{310}{27}$, so the minimal value for $f(x,y)$ is 2.

7. (25 points) Double Integrals over regular regions.

   (a) (10 points) Evaluate $\int_0^1 \int_{-x}^{x^3} x \, \mathrm{d}y \, \mathrm{d}x$.

   Answer:

   $\int_0^1 \int_{-x}^{x^3} x \, \mathrm{d}y \, \mathrm{d}x = \int_0^1 xy|_{y=-x}^{y=x^3} \, \mathrm{d}x = \int_0^1 x^4 + x^2 \, \mathrm{d}x = \frac{8}{15}$.

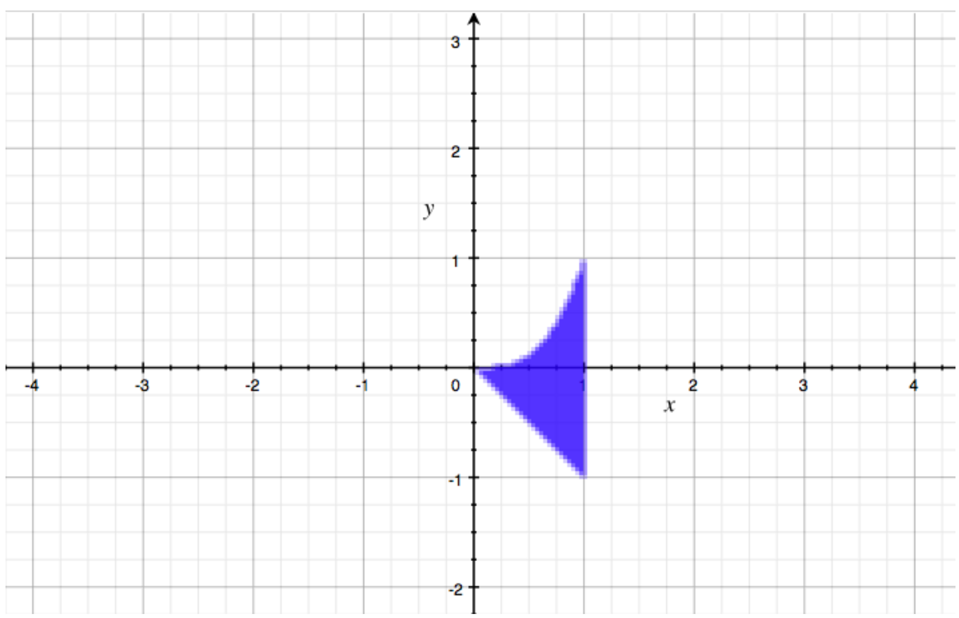
(b) (5 points) Graph the region of integration and describe it as a regular x region and a regular y region.

Answer:

$R = \{(x, y) | 0 \leq x \leq 1, \ -x \leq y \leq x^3\} = \{(x, y) | -1 \leq y \leq 1, \ h(y) \leq x \leq 1\}$, where

$$h(y) = \begin{cases} -y & -1 \leq y \leq 0 \\ \sqrt[3]{y} & 0 \leq y \leq 1 \end{cases}$$

The region is shown as follows.



(c) (10 points) Reverse the order of integration and evaluate the integral with the order reversed. Do you get the same result as part (a) ?

Answer:

$\int_{-1}^{1}\int_{h(y)}^{1} x\ \mathrm{d}x\ \mathrm{d}y = \int_{-1}^{0}\int_{-y}^{1} x\ \mathrm{d}x\ \mathrm{d}y + \int_{0}^{1}\int_{\sqrt[3]{y}}^{1} x\ \mathrm{d}x\ \mathrm{d}y.$

$\int_{-1}^{0}\int_{-y}^{1} x\ \mathrm{d}x\ \mathrm{d}y = \int_{-1}^{0} \frac{1}{2}x^2|_{x=-y}^{x=1}\ \mathrm{d}y = \int_{-1}^{0} \frac{1}{2} - \frac{1}{2}y^2\ \mathrm{d}y = (\frac{1}{2}y - \frac{1}{6}y^3)|_{y=-1}^{y=0} = \frac{1}{3},$

$\int_{0}^{1}\int_{\sqrt[3]{y}}^{1} x\ \mathrm{d}x\ \mathrm{d}y = \int_{0}^{1} \frac{1}{2}x^2|_{x=\sqrt[3]{y}}^{x=1}\ \mathrm{d}y = \int_{0}^{1} \frac{1}{2} - \frac{1}{2}y^{2/3}\ \mathrm{d}y = (\frac{1}{2}y - \frac{3}{10}y^{5/3})|_{y=0}^{y=1} = \frac{1}{5},$

Therefore, $\int_{-1}^{1}\int_{h(y)}^{1} x\ \mathrm{d}x\ \mathrm{d}y = \frac{1}{3} + \frac{1}{5} = \frac{8}{15}.$

The same result as part (a)!

8. (10 points) (*Extra Credits*) Evaluate $\int e^{3x} \sin 2x \, dx$.

Answer:

Choose $u = \sin 2x$ and $dv = e^{3x} dx$, then $du = 2\cos 2x \, dx$ and $v = e^{3x}/3$.

$$\int e^{3x} \sin 2x \, dx = \int u dv = uv - \int v du = \frac{1}{3} e^{3x} \sin 2x - \frac{2}{3} \int e^{3x} \cos 2x \, dx \tag{1}$$

Now consider the integral $\int e^{3x} \cos 2x \, dx$.

Choose $u = \cos 2x$ and $dv = e^{3x} dx$, then $du = -2\sin 2x \, dx$ and $v = e^{3x}/3$.

$$\int e^{3x} \cos 2x \, dx = \int u dv = uv - \int v du = \frac{1}{3} e^{3x} \cos 2x + \frac{2}{3} \int e^{3x} \sin 2x \, dx \tag{2}$$

By plugging Equation 2 back into Equation 1, we will have

$$\int e^{3x} \sin 2x \, dx = \frac{1}{3} e^{3x} \sin 2x - \frac{2}{3}\left(\frac{1}{3} e^{3x} \cos 2x + \frac{2}{3} \int e^{3x} \sin 2x \, dx\right),$$

which implies

$$\int e^{3x} \sin 2x \, dx = \frac{1}{13}(3e^{3x} \sin 2x - 2e^{3x} \cos 2x) + C.$$