Solutions to Homework 1 for credits

1 2.1.3

Let $X$ and $Y$ denote the respective outcomes with two fair dices are thrown. Let $U = \min\{X, Y\}$, $V = \max\{X, Y\}$, and $S = U + V$, $T = V - U$.

(a) Determine the conditional probability mass function for $U$ given $V = v$.
(b) Determine the joint mass function for $S$ and $T$.

Solutions:
(a) The idea for (a) is to consider different cases: $X = Y$, $X < Y$, $X > Y$.
For $u, v = 1, 2, \cdots, 6$, the joint p.m.f for $U$ and $V$ is

$$p_{U,V}(u,v) = \begin{cases} 0 & \text{if } u > v, \\ \frac{1}{36} & \text{if } u = v, \\ \frac{1}{18} & \text{if } u < v. \end{cases}$$

For $v = 1, 2, \cdots, 6$, the p.m.f for $V$ is

$$p_V(v) = \sum_{u=1}^{6} p_{U,V}(u,v) = \frac{1}{36} + \sum_{u=1}^{v-1} \frac{1}{18} = \frac{1}{36} + \frac{v-1}{18}.$$

Thus, the conditional p.m.f. for $u, v = 1, 2, \cdots, 6$ can be given by

$$p_{U|V}(u|v) = \frac{p_{U,V}(u,v)}{p_V(v)} = \begin{cases} 0 & \text{if } u > v, \\ \frac{1}{2v-1} & \text{if } u = v, \\ \frac{2}{2v-1} & \text{if } u < v. \end{cases}$$

(b) The idea for (b) is that you can solve $U$ and $V$ from $S$ and $T$, so then

$$p_{S,T}(s,t) = p_{U,V}\left(\frac{s-t}{2}, \frac{s+t}{2}\right) = \begin{cases} 0 & \text{if } t < 0, \\ \frac{1}{36} & \text{if } t = 0, \\ \frac{1}{18} & \text{if } t > 0. \end{cases}$$

2 2.3.4

Suppose $\xi_1, \xi_2, \cdots$ are independent and identically distributed random variables having mean $\mu$ and variance $\sigma^2$. Form the random sum $S_N = \xi_1 + \cdots + \xi_N$.

(a) Derive the mean and variance of $S_N$ when $N$ has a Poisson distribution with parameter $\lambda$. 

1
(b) Determine the mean and variance of $S_N$ when $N$ has a geometric distribution with mean $\lambda = \frac{1-p}{p}$.

(c) Compare the behaviors in (a) and (b) as $\lambda \to \infty$.

**Solutions:**

(a) Following 2.3.2, one can derive that $E[S_N] = \lambda \mu$ and $\text{Var}[S_N] = \lambda(\sigma^2 + \mu^2)$.

(b) Since $E[N] = \lambda = \frac{1-p}{p}$, we have $\frac{1}{p} = \lambda + 1$, then the variance for geometric distributed $N$ is $\text{Var}[N] = \frac{1-p}{p^2} = \lambda(\lambda + 1)$. Following the result from 2.3.2, one have $E[S_N] = \lambda \mu$ and $\text{Var}[S_N] = \lambda \sigma^2 + \mu^2 \lambda(\lambda + 1) = \lambda(\sigma^2 + \mu^2) + \mu^2 \lambda^2$.

(c) As $\lambda \to \infty$, the means for both cases are the same and grow linearly. The variance in case (b) grows faster than variance in case (a); variance in (a) grows linearly while variance in (b) grows quadratically.

3 2.4.7

Suppose that $X$ and $Y$ are independent random variables, each having the same exponential distribution with parameter $\alpha$. What is the conditional probability density function for $X$, given that $Z = X + Y = z$?

**Solutions:**

Using 1.4.4, one can have

$$f_Z(z) = \begin{cases} \alpha^2 z e^{-\alpha z} & \text{if } z \geq 0, \\ 0 & \text{if } z < 0. \end{cases}$$

Since $X$, $Y$ are independent exponential distribution with parameter $\alpha$,

$$f_X(x) = f_Y(x) = \begin{cases} \alpha e^{-\alpha x} & \text{if } x \geq 0, \\ 0 & \text{if } x < 0. \end{cases}$$

Then by definition of conditional probability density, we get

$$f_{X|Z}(x|z) = \frac{f_{X,Z}(x,z)}{f_Z(z)} = \frac{f_{X,Y}(x,z-x)}{f_Z(z)} = \begin{cases} \frac{(\alpha e^{-\alpha x})(\alpha e^{-\alpha(z-x)})}{\alpha^2 \alpha e^{-\alpha z}} = \frac{1}{z} & \text{if } 0 \leq x \leq z, \\ 0 & \text{otherwise}, \end{cases}$$

i.e. $X \sim \text{Unif}[0,z]$, conditioned on $Z = z$.

4 2.4.8

Let $X$ and $Y$ have the normal density given in Chapter 1 (1.47) or the following Eq (1).

$$f_{X,Y}(x,y) = \frac{1}{2\pi \sigma_X \sigma_Y \sqrt{1-\rho^2}} \exp\left(-\frac{1}{2(1-\rho^2)} \left[ \frac{(x-\mu_X)^2}{\sigma_X^2} - 2\rho \frac{(x-\mu_X)(y-\mu_Y)}{\sigma_X \sigma_Y} + \frac{(y-\mu_Y)^2}{\sigma_Y^2} \right] \right)$$

(1)

Show that the conditional density function for $X$, given that $Y = y$, is normal with moments

$$\mu_{X|Y} = \mu_X + \rho \frac{\sigma_X}{\sigma_Y} (y - \mu_Y)$$
and

\[ \sigma_{X|Y} = \sigma_X \sqrt{1 - \rho^2}. \]

Solutions:
Applying conditional density function definition on these joint p.d.f and p.d.f:

\[ f_{X,Y}(x, y) = \frac{1}{2\pi \sigma_X \sigma_Y \sqrt{1 - \rho^2}} \exp\left\{ -\frac{1}{2(1 - \rho^2)} \left[ (\frac{x - \mu_X}{\sigma_X})^2 - 2\rho \left( \frac{x - \mu_X}{\sigma_X} \right) \left( \frac{y - \mu_Y}{\sigma_Y} \right) + \left( \frac{y - \mu_Y}{\sigma_Y} \right)^2 \right] \right\}, \]

\[ f_Y(y) = \frac{1}{\sqrt{2\pi \sigma_Y^2}} \exp\left\{ -\frac{1}{2} \left( \frac{y - \mu_Y}{\sigma_Y} \right)^2 \right\}, \]

one can get the conditional p.d.f.

\[ f_{X|Y}(x|y) = \frac{f_{X,Y}(x, y)}{f_Y(y)}, \]

\[ = \frac{1}{\sqrt{2\pi \sigma_X^2 (1 - \rho^2)}} e^{-\frac{1}{2(1 - \rho^2)} \left[ (\frac{x - \mu_X}{\sigma_X})^2 - 2\rho \left( \frac{x - \mu_X}{\sigma_X} \right) \left( \frac{y - \mu_Y}{\sigma_Y} \right) + \left( \frac{y - \mu_Y}{\sigma_Y} \right)^2 \right]}, \]

\[ = \frac{1}{\sqrt{2\pi \sigma_X^2 (1 - \rho^2)}} \exp\left\{ -\frac{1}{2(1 - \rho^2)} \left( \frac{x - \mu_X}{\sigma_X} - \rho \frac{(y - \mu_Y)}{\sigma_Y} \right)^2 \right\}, \]

\[ = \frac{1}{\sqrt{2\pi \sigma_X^2 (1 - \rho^2)}} \exp\left\{ -\frac{1}{2} \left( \frac{x - (\mu_X + \rho \frac{\sigma_X}{\sigma_Y} (y - \mu_Y))}{\sigma_X^2 (1 - \rho^2)} \right) \right\}. \]

Based on the expression of the conditional p.d.f., it’s clear that \( \sigma_{X|Y} = \sigma_X \sqrt{1 - \rho^2} \) and \( \mu_{X|Y} = \mu_X + \rho \frac{\sigma_X}{\sigma_Y} (y - \mu_Y). \)

5 2.5.4

Let \( \xi_1, \xi_2, \ldots \) be independent Bernoulli random variables with parameter \( p, \) \( 0 < p < 1. \) Show that \( X_0 = 1 \) and \( X_n = p^{-n} \xi_1 \cdots \xi_n, \) \( n = 1, 2, \ldots, \) defines a nonnegative martingale. What is the limit of \( X_n \) as \( n \to \infty? \)

Solutions:

\[
\begin{align*}
X_n &= p^{-n} \xi_1 \cdots \xi_n, \\
X_0 &= 1,
\end{align*}
\]

can be rewritten as

\[
\begin{align*}
X_{n+1} &= p^{-1} \xi_{n+1} X_n, \\
X_0 &= 1.
\end{align*}
\]

The first definition can tell you that \( X_n \) is always non-negative. Use the second definition to check this is a martingale!
For martingale property:

\[ E[X_{n+1}|X_0, \ldots, X_n] = E[X_{n+1}|X_n] = p^{-1}E[\xi_{n+1}]X_n = X_n. \]

Since it satisfies martingale properties, we can further have \( E[X_n] = E[X_0] = 1 \). Thus, \( E[|X_n|] = E[X_n] < \infty \).

From the first definition, we have that \( X_n > 0 \) only when \( \xi_1 = \xi_2 = \cdots = \xi_n = 1 \), otherwise, \( X_n = 0 \). Then we’ll have \( P(X_n > 0) = p^n \) and therefore \( \lim_{n \to \infty} P(X_n > 0) = \lim_{n \to \infty} p^n = 0 \), which provides that \( \lim_{n \to \infty} X_n = 0 \) in probability.