Lecture on July 10th, 2019
Some Special Markov Chains

1 Two-State Model - See Chap 3.5.1

- Let \( \{X_n\} \) be a MC with

\[
P = \begin{pmatrix}
1 - a & a \\
b & 1 - b
\end{pmatrix}
\]

where \( 0 < a, b < 1 \).

- If \( 1 - a = b \), \( X_n \)'s are i.i.d.

- If \( 1 - a \neq b \), distribution of \( X_{n+1} \) depends on the value of \( X_n \).
  - Claim (Check by induction):
    \[
P(n) = \frac{1}{a + b} \begin{pmatrix} b & a \\ b & a \end{pmatrix} + \frac{(1 - a - b)^n}{a + b} \begin{pmatrix} a & -a \\ -b & b \end{pmatrix}.
    \]
    - As \( n \to \infty \),
      \[
      \lim_{n \to \infty} P(n) = \frac{1}{a + b} \begin{pmatrix} b & a \\ b & a \end{pmatrix}.
      \]
  - The long-time limiting distribution doesn’t depend on the initial distribution!

2 Markov Chains Defined by Random Variables

- Ex: independent r.v. \( X_n = \xi_n \), where \( \xi_n \)'s are i.i.d. In this case, the transition matrix doesn’t depend on initial states.

- Ex: Successive Maxima. \( X_n = \max\{\xi_1, \xi_2, \ldots, \xi_n\} \) for \( n \geq 1 \) and \( X_0 = 0 \). Check this is a M.C.

- Ex: Partial Sums. \( X_n = \sum_{i=1}^{n} \xi_i \), for \( n \geq 1 \) and \( X_0 = 0 \). Check that this is a M.C.

3 Gambler’s Ruin Problem

- Assume 0 and N are two absorbing states. Let \( u_i \) be the probability of reaching state 0 starting at fortune \( i \).
• By first step analysis,

\[
\begin{aligned}
&u_0 = 1, u_N = 0, \\
&u_i = pu_{i+1} + qu_{i-1}, \quad i = 1, 2, \cdots, N - 1
\end{aligned}
\]