Lecture on July 10th, 2019
Some Special Markov Chains

1 Two-State Model - See Chap 3.5.1

- Let $\{X_n\}$ be a MC with
$$P = \begin{pmatrix} 1-a & a \\ b & 1-b \end{pmatrix}$$
where $0 < a, b < 1$.

- If $1 - a = b$, X_n's are i.i.d.
- If $1 - a \neq b$, distribution of X_{n+1} depends on the value of X_n.
  - Claim (Check by induction):
$$P^{(n)} = \frac{1}{a+b}\begin{pmatrix} b & a \\ b & a \end{pmatrix} + \frac{(1-a-b)^n}{a+b}\begin{pmatrix} a & -a \\ -b & b \end{pmatrix}.$$
  - As $n \to \infty$,
$$\lim_{n\to\infty} P^{(n)} = \frac{1}{a+b}\begin{pmatrix} b & a \\ b & a \end{pmatrix}.$$
  - The long-time limiting distribution doesn't depend on the initial distribution!

2 Markov Chains Defined by Random Variables

- Ex: independent r.v. $X_n = \xi_n$, where ξ_n's are i.i.d. In this case, the transition matrix doesn't depends on initial states.
- Ex: Successive Maxima. $X_n = \max\{\xi_1, \xi_2, \cdots, \xi_n\}$ for $n \geq 1$ and $X_0 = 0$. Check this is a M.C.
- Ex: Partial Sums. $X_n = \sum_{i=1}^{n} \xi_i$, for $n \geq 1$ and $X_0 = 0$. Check that this is a M.C.

3 Gambler's Ruin Problem

- Assume 0 and N are two absorbing states. Let u_i be the probability of reaching state 0 starting at fortune i.

- By first step analysis,

$$
\begin{cases}
u_0 = 1, u_N = 0, \\
u_i = pu_{i+1} + qu_{i-1}, \quad i = 1, 2, \cdots, N-1
\end{cases}
$$