Introduction

In this lab we'll continue our study of relatively simple two-dimensional models. We'll learn how to check the stability of fixed points, and consider some examples where the Hartman-Grobman theorem does not apply. We'll also look for bifurcations. To do so, we'll again use the software package XPPAUT. You've already downloaded and installed XPPAUT last week. See instructions from the previous lab if not.

Hartman-Grobman for a simple system

Let's start by considering the following system of differential equations:

$$\dot{x} = xy \tag{1}$$
$$\dot{y} = -y - x^2 \tag{2}$$

First, we'll "solve" this system by hand in lab. We'll find the fixed points, and check the stability of these fixed points. Does the Hartman-Grobman theorem hold in this case (i.e., is there a zero eigenvalue)? Second, we'll study this system numerically. To do so, we'll use XPPAUT and do the following:

- Write an *.ode* file for this system.
- Compute numerical solutions to the model and plot in x, y space.
- Plot the scaled direction field in the x, y space.
- Find the fixed point(s) numerically and check stability (through examination of eigenvalues).

Using XPPAUT to make bifurcation diagrams

In class (and today in lab) we'll eventually construct a bifurcation diagram for the FitzHugh-Nagumo model. We started last time by computing (in Mathematica) the curve of fixed points $\bar{V}$ as a function of parameter I. We also determined that the stability of these fixed points changed at particular values of I. We've already seen how XPPAUT is a useful tool for computing numerical solutions to ODEs, plotting direction fields, and finding nullclines. This software is also useful for creating bifurcation diagrams. To do so, XPPAUT interfaces with the software package AUTO — software for continuation and bifurcation problems in ordinary differential equations. We won't go into the details of how AUTO works. Instead, we'll see how we can use XPPAUT to quickly draw a bifurcation diagram for the FitzHugh-Nagumo model.

Specifically here's what we'll do:

1. Open our FitzHugh-Nagumo model system in XPPAUT.
2. Set $I = 0$ and compute a numerical solution to the model. The dynamics should approach a fixed point.
3. Run AUTO and find the curve of fixed points for this model. We'll see that we can identify the fixed point stability and the bifurcations.

Challenge Problems (Please answer three.)

1. "Solve" the two systems below. More specifically: First find the fixed points and check their stability. Try to do this by hand, but also feel free to use XPPAUT. Determine for each fixed point whether the Hartman-Grobman theorem holds (i.e., is there a zero eigenvalue?). Compare the results of your stability analysis with the full nonlinear dynamics near the fixed point investigated through numerical simulations using XPPAUT. Do your numerical simulations show that the fixed point is stable or unstable?

$$\dot{x} = xy - y \tag{3}$$
$$\dot{y} = xy - x \tag{4}$$

$$\dot{x} = x^2 - xy \tag{5}$$
$$\dot{y} = -y + x^2 \tag{6}$$

2. Use XPPAUT to study the dynamics of the Morris-Lecar model. You decide what "study" means. I'll recommend investigating how the dynamics change as parameter i is varied within the range $0 < i < 400$. You can read about the model here:

   `http://www.scholarpedia.org/article/Morris-Lecar_model`

   Here's code for the ODE file:

```
# Morris-Lecar reduced model

# Set the fixed parameters.
vk=-84
vl=-60
vca=120
gk=8
gl=2
gca=4
c=20
v1=-1.2
v2=18
v3=2
v4=30
phi=.04

# The model equations
dv/dt=(i+gl*(vl-v)+gk*w*(vk-v)+gca*minf(v)*(vca-v))/c
dw/dt=lamw(v)*(winf(v)-w)

# where
minf(v)=.5*(1+tanh((v-v1)/v2))
winf(v)=.5*(1+tanh((v-v3)/v4))
lamw(v)=phi*cosh((v-v3)/(2*v4))

# Set the adjustable parameter.
param i=0

# Set the initial conditions.
v(0)=-60.899 w(0)=0.014873

#XPPAUT parameters
@ total=2000
@ parmin=-10, parmax=400, nmax=1000
@ maxstor=500000

done
```

3. Use XPPAUT to study the dynamics of the Wilson-Cowan model for parameter p, $-5 \leq p \leq 0$. You decide what "study" means. Here's the code for the ODE file, in which p is the only adjustable parameter (all the others are fixed):

```
# The Wilson-Cowan Model

# Set the fixed parameters
a=16
b=12
c=16
d=5
q=-4
```

```
# the wilson-cowan equations
u'=-u+f(a*u-b*v+p)
v'=-v+f(c*u-d*v+q)
f(u)=1/(1+exp(-u))

# Set the adjustable parameter.
param p=-5

# Set some options for a nice plot.
@ xp=u,yp=v,xlo=-.125,ylo=-.125,xhi=1,yhi=1

# Set some options for AUTO
@ parmin=-5, parmax=5, dsmax=0.05
@ maxstor=500000
done
```

4. Study a neural system of your choice. Follow any path you choose to "study" the model. Some places to look for pre-existing models:

```
http://www.math.pitt.edu/~bard/bardware/tut/newstyle.html
http://www.math.pitt.edu/~bard/xpp/odes/
```

5. For the FitzHugh-Nagumo model use the Poincare-Bendixson theorem to show that a periodic orbit must exist (for an appropriate choice of I). HINT: Consider the V, W-phase plane and determine a bounded region along with the direction field always points inward. Perhaps consider a rectangular region with corners at the nullclines . . .

6. Make up your own problem. Make it really interesting!