Counting modular forms with fixed mod-p Galois representation and Atkin-Lehner-at-p eigenvalue

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Boston University

2023 JMM

AMS Special Session on Women in Automorphic Forms I

January 6, 2023

1. Modular forms of level p

Fix a prime $p \geq 5$. (Can also add tame level N, omitted here.)

$$S_k(p) := \text{ space of cusp forms of weight } k \text{ and level } \Gamma_0(p)$$
$$d_k := \dim S_k(p)$$

Dimension formulas means d_k is well known.

In particular, d_k grows linearly in k:

$$d_k \sim \frac{(p+1)k}{12} \text{ as } k \to \infty.$$

2. Atkin-Lehner operator W_p splits $S_k(p)$

The Atkin-Lehner operator W_p is an involution that acts on S_k. So

$$S_k(p) = S_k(p)^+ \oplus S_k(p)^-,$$

where W_p acts as $+1$ on $S_k(p)^+$ and as -1 on $S_k(p)^-$.

- What is the split in dimension?

Let $d_k^{\pm} := \dim S_k(p)^{\pm}$. Since d_k is known, study

$$\Delta_k := d_k^+ - d_k^-.$$

Note: $\Delta_k = \mathrm{Tr}\left(W_p | S_k(p)\right)$.

3. Data!

$p = 5$

k	d_k^+	d_k^-
2	0	0
4	1	0
6	0	1
8	2	1
10	1	2
12	3	2
14	2	3
16	4	3
18	3	4
20	5	4
22	4	5
24	6	5
26	5	6

$\Delta_k = \pm 1$

$p = 23$

k	d_k^+	d_k^-
2	0	2
4	4	1
6	3	6
8	8	5
10	7	10
12	12	9
14	11	14
16	16	13
18	15	18
20	20	17
22	19	22
24	24	21
26	23	26

$\Delta_k = \pm 3$

$p = 101$

k	d_k^+	d_k^-
2	1	7
4	16	9
6	17	24
8	33	26
10	34	41
12	50	43
14	51	58
16	67	60
18	68	75
20	84	77
22	85	92
24	101	94
26	102	109

$\Delta_k = \pm 7$

4. $|\Delta_k|$ is basically a class number!

Theorem (Fricke, Yamauchi, Helfgott, Wakatsuki, Martin...)

$$\Delta_k = (-1)^{k/2}\frac{\#\mathrm{FP}}{2}$$

(correction if $k = 2^*$: add 1)

- Here $\#\mathrm{FP}$ is the number of fixed points of the geometric Atkin-Lehner involution on the modular curve $X_0(p)$.
- The moduli interpretation for $X_0(p)$ relates this number to elliptic curves with CM by $\sqrt{-p}$.
- So $\#\mathrm{FP} = \begin{cases} h\big(\mathbb{Q}(\sqrt{-p})\big) & \text{if } p \equiv 1 \bmod 4, \\ h\big(\mathbb{Q}(\sqrt{-p})\big) + h\big(\mathbb{Z}[\sqrt{-p}]\big) & \text{if } p \equiv 3 \bmod 4. \end{cases}$

Example: For $p = 5$, $h\big(\mathbb{Q}(\sqrt{-p})\big) = 2$ and $\Delta_k = \pm 1$.
For $p = 101$, $h\big(\mathbb{Q}(\sqrt{-p})\big) = 14$ and $\Delta_k = \pm 7$.

Corollary

$$\Delta_{k+2} = -\Delta_k \qquad \textit{for } k \geq 2^*$$

5. Refine for congruences between modular forms

(Work with $\mathbb{Q}_p$ or $\bar{\mathbb{Q}}_p$ coefficients here.)

Spaces $S_k(p)$ have action of Hecke operators. Here suffices to consider T_ℓ for $\ell \neq p$. Can find basis of eigenforms for T_ℓ.

Eigenvalues of T_ℓ are algebraic integers, so consider them mod p. Systems of mod-p Hecke eigenvalues τ correspond to Galois representations $\rho_\tau : G_\mathbb{Q} \to \mathrm{GL}_2(\bar{\mathbb{F}}_p)$, with $\tau(\ell) = \mathrm{Tr}\,\rho_\tau(\mathrm{Frob}_\ell)$. Set

$S_k(p)_\tau :=$ span of eigenforms with mod-p Hecke eigensystem τ.

The Atkin-Lehner involution W_p commutes with the T_ℓ, so again

$$S_k(p)_\tau = S_k(p)_\tau^+ \oplus S_k(p)_\tau^-.$$

with corresponding dimensions

$$d_{k,\tau} = d_{k,\tau}^+ + d_{k,\tau}^-.$$

Again $d_{k,\tau}$ grows linearly with k (Jochnowitz, Bergdall-Pollack); set

$$\Delta_{k,\tau} := d_{k,\tau}^+ - d_{k,\tau}^-.$$

6. Twisting!

The Hecke eigensystem τ can only appear in weight k if $\det \rho_\tau = \omega^{k-1}$, where ω is the mod-p cyclotomic character.

In other words, τ determines k modulo $p-1$.

But we can move between weights by the θ operator: τ becomes $\tau[1]$ with $\tau[1](\ell) = \ell\tau(\ell)$. On the Galois side, this is twisting by ω.

If τ can appear in weight k, then

$\tau[1] \longleftrightarrow \rho_\tau \otimes \omega$ can appear in weight $k+2$

$\tau[2] \longleftrightarrow \rho_\tau \otimes \omega^2$ can appear in weight $k+4$

$\dots$

$\tau[\frac{p-1}{2}] \longleftrightarrow \rho_\tau \otimes \left(\frac{\cdot}{p}\right)$ can appear in weight $k+(p-1)$, or in weight k

$\dots$

$\tau[p-1] \longleftrightarrow \rho_\tau$ can appear in weight $k+2(p-1)$, or in weight k

7. More data!

$p = 5, N = 23$ Dimension splits $(d^{+}_{k,\tau},\ d^{-}_{k,\tau})$ in weight k for τ.

$k \backslash \tau$	σ	$\sigma[1]$	$\sigma[2]$	$\sigma[3]$
2	$(3,2)$	—	$(0,0)$	—
4	—	$(2,3)$	—	$(0,0)$
6	$(5,5)$	—	$(3,2)$	—
8	—	$(5,5)$	—	$(2,3)$
10	$(8,7)$	—	$(5,5)$	—
12	—	$(7,8)$	—	$(5,5)$
14	$(10,10)$	—	$(8,7)$	—
16	—	$(10,10)$	—	$(7,8)$
18	$(13,12)$	—	$(10,10)$	—
20	—	$(12,13)$	—	$(10,10)$
22	$(15,15)$	—	$(13,12)$	—
24	—	$(15,15)$	—	$(12,13)$

8. First main result

Theorem (Anni–Ghitza–M.) *(Recall $p \geq 5$; tame level N ok)*

$$\Delta_{k+2,\tau[1]} = -\Delta_{k,\tau} \qquad \text{for } k \geq 2^*$$

Theorem follows from an up-to-semisimplification isomorphism between two mod-p Hecke modules.

Which mod-p Hecke modules?
Space $S_{k-p+1}(Np, \mathbb{F}_p)$ embeds into $S_k(Np, \mathbb{F}_p)$ in a Hecke equivariant way by multiplication by Hasse invariant E_{p-1}.

Corresponding graded module is $W_k(Np)$.

- (Jochnowitz, Serre, Robert) $W_{k+p+1}(N) \simeq W_k(N)[1]$
- (Bergdall–Pollack, AGM) $W_{k+2}(Np)^{\mathrm{ss}} \simeq W_k(Np)[1]^{\mathrm{ss}}$

9. Second main result

We construct a refinement of $W_k(Np)$: given two signs ε, η, define

$$W_k(Np)^{\varepsilon,\eta} := S_k(Np, \mathbb{F}_p)^{\varepsilon} / S_{k-p+1}(Np, \mathbb{F}_p)^{\eta}.$$

Theorem (Anni–Ghitza–M.)

For any $k \geq (p+1)^*$ and any signs ε, η in $\{\pm 1\}$, we have

$$W_{k+2}^{\varepsilon,\eta}(Np)^{\mathrm{ss}} \simeq W_k^{-\varepsilon,-\eta}(Np)[1]^{\mathrm{ss}}.$$

Technical details

Define $S_k(Np, \mathbb{F}_p)^{\pm} := \big(S_k(Np, \mathbb{Z}_p) \cap S_k(Np, \mathbb{Q}_p)^{\pm}\big) \otimes \mathbb{F}_p$. Then $S_{k-p+1}(Np, \mathbb{F}_p)^{\eta}$ embeds into $S_k(Np, \mathbb{F}_p)^{\varepsilon}$ by multiplication by the Atkin-Lehner "stabilization" $E_{p-1}^{\varepsilon/\eta}$ of E_{p-1}, where

$$E_{p-1}^{\pm} := E_{p-1} \pm p^{(p-1)/2} E_{p-1}(pz).$$

10. Method of proof: algebra lemma + trace formula

To establish isomorphism of semisimplified mod-p Hecke modules, we develop new technique: deeper congruences with trace formula.

Lemma (AGM; refines Brauer–Nesbitt for $\mathbb{Z}_p[T]$)

Let M, N be rank-d free $\mathbb{Z}_p$-modules with linear action of T. Then

$$\overline{M}^{\mathrm{ss}} \simeq \overline{N}^{\mathrm{ss}} \iff \mathrm{Tr}(T^n \mid M) \equiv \mathrm{Tr}(T^n \mid N) \mod p^{1+v_p(n)}$$

for every $1 \leq n \leq d$. Here p can be any prime!

Here $\overline{M}^{\mathrm{ss}}$ is the semisimplification of $\mathbb{F}_p[T]$-module $M \otimes \mathbb{F}_p$.

Example (of Goldilocks titration)

Set $M := \mathbb{Z}_p^{\oplus p}$ with T acting by $\alpha \in \mathbb{Z}_p$, so $\mathrm{Tr}(T^n|M) = p\alpha^n$.

- Knowing $p\alpha^n$ in $\mathbb{Z}_p$ identifies α in $\mathbb{Z}_p$ — too much!
- Knowing $p\alpha^n = 0$ in $\mathbb{F}_p$ tells us nothing — too little!
- But $p\alpha^p$ mod p^2 identifies α^p (and so α) mod p — just right!

11. Remarks about main theorem

Recall main theorem.

Theorem (AGM)

$$\Delta_{k+2,\tau[1]} = -\Delta_{k,\tau} \qquad \textit{for } k \geq 2^*$$

Remarks

- As a corollary, uneven splits always come from weight 2.
- Quite generally, uneven splits come from p-new forms (p-old forms always come in $\pm$ Atkin-Lehner pairs).
- No τ can appear p-newly in weight 2 with both $\pm$ signs. (In weight k a p-new form has $a_p = \pm p^{\frac{k-2}{2}}$, with the sign determined by the Atkin-Lehner eigenvalue. Therefore in weight 2 we can see the sign mod p from $a_p = \pm 1$.)
- Thus $\Delta_{k,\tau} = 0$ unless $\tau[\frac{2-k}{2}]$ appears p-newly in weight 2.

12. Even more data!

$p = 5,\ N = 23$ Up to twist, there are 7 Galois orbits of eigensystems that appear.

$k \backslash \tau$	e	$e[2]$	σ	$\sigma[2]$	t	$t[2]$	s	$s[2]$	$f,\ f[2]$ $g,\ g[2]$ $h,\ h[2]$
2	$(0,0)$	$(0,0)$	$(3,2)$	$(0,0)$	$(2,0)$	$(0,0)$	$(0,1)$	$(0,0)$	$(0,0)$
4	$(2,1)$	$(0,0)$	$(2,3)$	$(0,0)$	$(0,2)$	$(0,0)$	$(1,0)$	$(0,0)$	$(1,1)$
6	$(1,2)$	$(1,1)$	$(3,2)$	$(5,5)$	$(2,0)$	$(2,2)$	$(0,1)$	$(1,1)$	$(1,1)$
8	$(2,1)$	$(3,3)$	$(2,3)$	$(5,5)$	$(0,2)$	$(2,2)$	$(1,0)$	$(1,1)$	$(2,2)$
10	$(2,3)$	$(3,3)$	$(8,7)$	$(5,5)$	$(4,2)$	$(2,2)$	$(1,2)$	$(1,1)$	$(2,2)$
12	$(5,4)$	$(3,3)$	$(7,8)$	$(5,5)$	$(2,4)$	$(2,2)$	$(2,1)$	$(1,1)$	$(3,3)$
14	$(4,5)$	$(4,4)$	$(8,7)$	$(10,10)$	$(4,2)$	$(4,4)$	$(1,2)$	$(2,2)$	$(3,3)$
16	$(5,4)$	$(6,6)$	$(7,8)$	$(10,10)$	$(2,4)$	$(4,4)$	$(2,1)$	$(2,2)$	$(4,4)$
18	$(5,6)$	$(6,6)$	$(13,12)$	$(10,10)$	$(6,4)$	$(4,4)$	$(2,3)$	$(2,2)$	$(4,4)$
20	$(8,7)$	$(6,6)$	$(12,13)$	$(10,10)$	$(4,6)$	$(4,4)$	$(3,2)$	$(2,2)$	$(5,5)$

- e is the Eisenstein eigensystem in weight 2: $e(\ell) = 1 + \ell$
- s is a $\mathbb{F}_{5^4}$-Galois orbit of 4 eigensystems; h is an $\mathbb{F}_{5^3}$-orbit of 3 eigensystems
- σ has Serre weight 2 (peu ramifié); t and s have Serre weight 6 (très ramifié); f, g, h have Serre weight 4