$p = 5$

k	d_k^+	d_k^-
2	0	0
4	1	0
6	0	1
8	2	1
10	1	2
12	3	2
14	2	3
16	4	3
18	3	4
20	5	4
22	4	5
24	6	5
26	5	6
28	7	6
30	6	7

$\Delta_k = \pm 1$

$p = 23$

k	d_k^+	d_k^-
2	0	2
4	4	1
6	3	6
8	8	5
10	7	10
12	12	9
14	11	14
16	16	13
18	15	18
20	20	17
22	19	22
24	24	21
26	23	26
28	28	25
30	27	30

$\Delta_k = \pm 3$

$p = 59$

k	d_k^+	d_k^-
2	0	5
4	10	4
6	9	15
8	20	14
10	19	25
12	30	24
14	29	35
16	40	34
18	39	45
20	50	44
22	49	55
24	60	54
26	59	65
28	70	64
30	69	75

$\Delta_k = \pm 6$

$p = 101$

k	d_k^+	d_k^-
2	1	7
4	16	9
6	17	24
8	33	26
10	34	41
12	50	43
14	51	58
16	67	60
18	68	75
20	84	77
22	85	92
24	101	94
26	102	109
28	118	111
30	119	126

$\Delta_k = \pm 7$

$$d_k^{\pm} = \dim S_k(p)^{\pm} \qquad \Delta_k = d_k^+ - d_k^-$$

$\boxed{p = 5, N = 23}$

Table of dimension splits $(d^+_{k,\bar{f}},\ d^-_{k,\bar{f}})$ in weight k for $\bar{f}$.

$k \backslash \bar{f}$	$\bar{g}$	$\bar{g}[1]$	$\bar{g}[2]$	$\bar{g}[3]$
2	$(3,2)$	—	$(0,0)$	—
4	—	$(2,3)$	—	$(0,0)$
6	$(5,5)$	—	$(3,2)$	—
8	—	$(5,5)$	—	$(2,3)$
10	$(8,7)$	—	$(5,5)$	—
12	—	$(7,8)$	—	$(5,5)$
14	$(10,10)$	—	$(8,7)$	—
16	—	$(10,10)$	—	$(7,8)$
18	$(13,12)$	—	$(10,10)$	—
20	—	$(12,13)$	—	$(10,10)$
22	$(15,15)$	—	$(13,12)$	—
24	—	$(15,15)$	—	$(12,13)$

Here $d^{\pm}_{k,\bar{f}} = \dim S_{k,\bar{f}}(p)^{\pm}$ is the multiplicity of $\bar{f}$ in $S_k(p)$ with Atkin-Lehner-at-p sign $\pm$.

$\boxed{p = 5, N = 23}$

Table of dimension splits $(d^+_{k,\bar f},\ d^-_{k,\bar f})$ in weight k for $\bar f$.

$k\backslash\bar f$	$\bar g$	$\bar g[1]$	$\bar g[2]$	$\bar g[3]$
2	$(3,2)$	—	$(0,0)$	—
4	—	$(2,3)$	—	$(0,0)$
6	$(5,5)$	—	$(3,2)$	—
8	—	$(5,5)$	—	$(2,3)$
10	$(8,7)$	—	$(5,5)$	—
12	—	$(7,8)$	—	$(5,5)$
14	$(10,10)$	—	$(8,7)$	—
16	—	$(10,10)$	—	$(7,8)$
18	$(13,12)$	—	$(10,10)$	—
20	—	$(12,13)$	—	$(10,10)$
22	$(15,15)$	—	$(13,12)$	—
24	—	$(15,15)$	—	$(12,13)$

Here $d^{\pm}_{k,\bar f} = \dim S_{k,\bar f}(p)^{\pm}$ is the multiplicity of $\bar f$ in $S_k(p)$ with Atkin-Lehner-at-p sign $\pm$.

$\boxed{p = 5,\ N = 23}$ Full data. In weight k for $\bar{f}$ the entry is $\left(d^{+}_{k,\bar{f}[\frac{k-2}{2}]},\ d^{-}_{k,\bar{f}[\frac{k-2}{2}]}\right)$.

$k\backslash\bar{f}$	$\bar{e}$	$\bar{e}'$	$\bar{g}$	$\bar{g}'$	$\bar{t}$	$\bar{t}'$	$\bar{s}\times 4$	$\bar{s}'\times 4$	$\bar{a},\bar{a}';\bar{b},\bar{b}'$; $\bar{c}\times 3,\bar{c}'\times 3$	Total
2	$(0,0)$	$(0,0)$	$(3,2)$	$(0,0)$	$(2,0)$	$(0,0)$	$(0,1)$	$(0,0)$	$(0,0)$	$(5,6)$
4	$(2,1)$	$(0,0)$	$(2,3)$	$(0,0)$	$(0,2)$	$(0,0)$	$(1,0)$	$(0,0)$	$(1,1)$	$(18,16)$
6	$(1,2)$	$(1,1)$	$(3,2)$	$(5,5)$	$(2,0)$	$(2,2)$	$(0,1)$	$(1,1)$	$(1,1)$	$(28,30)$
8	$(2,1)$	$(3,3)$	$(2,3)$	$(5,5)$	$(0,2)$	$(2,2)$	$(1,0)$	$(1,1)$	$(2,2)$	$(42,40)$
10	$(2,3)$	$(3,3)$	$(8,7)$	$(5,5)$	$(4,2)$	$(2,2)$	$(1,2)$	$(1,1)$	$(2,2)$	$(52,54)$
12	$(5,4)$	$(3,3)$	$(7,8)$	$(5,5)$	$(2,4)$	$(2,2)$	$(2,1)$	$(1,1)$	$(3,3)$	$(66,64)$
14	$(4,5)$	$(4,4)$	$(8,7)$	$(10,10)$	$(4,2)$	$(4,4)$	$(1,2)$	$(2,2)$	$(3,3)$	$(76,78)$
16	$(5,4)$	$(6,6)$	$(7,8)$	$(10,10)$	$(2,4)$	$(4,4)$	$(2,1)$	$(2,2)$	$(4,4)$	$(90,88)$
18	$(5,6)$	$(6,6)$	$(13,12)$	$(10,10)$	$(6,4)$	$(4,4)$	$(2,3)$	$(2,2)$	$(4,4)$	$(100,102)$
20	$(8,7)$	$(6,6)$	$(12,13)$	$(10,10)$	$(4,6)$	$(4,4)$	$(3,2)$	$(2,2)$	$(5,5)$	$(114,112)$

Two twists of $\bar{f}$ can appear in any given weight: $\bar{f}$ and its quadratic twist $\bar{f}' = \bar{f}[2]$

- $\bar{e}$ is the Eisenstein form in weight 2: $a_\ell(\bar{e}) = 1 + \ell$ for prime $\ell \neq p$
- $\bar{g}$ is a peu ramifié form, appearing in weight 2 here
- $\bar{t}$ is a très ramifié form, here appearing in weight 2
- $\bar{s}$ is an $\mathbb{F}_{5^4}$-Galois orbit of 4 très ramifié forms appearing in weight 2
- $\bar{a}, \bar{b}, \bar{c}$ are locally reducible, globally irreducible forms; c is an $\mathbb{F}_{5^3}$-orbit of 3 forms

Example for Expected Theorem

Take $p=5$ and $k=54$. Since $2p^2=50<k$, we can take $m=3$. We use $\ell=2$.

k,ε	$\overline{\text{charpoly}}\big(T_2 \mid S_k(5)^{\text{new},\varepsilon}\big)$ in $(\mathbb{Z}/125\mathbb{Z})[x]$
$54,+$	$x^8+10x^7+19x^6+80x^5+101x^4+5x^3+24x^2+60x+66$
$54,-$	$x^9+113x^8+49x^7+37x^6+91x^5+33x^4+39x^3+32x^2+121x+48$
$4,+$	$x+4$, so $E^-=x+4\cdot 2^{25}=x+103$
$4,-$	$1=E^+$

Then $\dfrac{\overline{\text{charpoly}}\big(T_2 \mid S_{54}(5)^{\text{new},+}\big)}{1}=\dfrac{\overline{\text{charpoly}}\big(T_2 \mid S_{54}(5)^{\text{new},-}\big)}{x+103}$, as expected.

On the other hand, here is the complete list of $\mathcal{L}$-invariant valuations for $S_{54}(5)^{\text{new}}$:

$$-2,\ -3,\ -3,\ ,-5,\ -5,\ -8,\ -8,\ -10,\ -10,\ -11,\ -11,\ -12,\ -12,\ -14,\ -14,\ -18,\ -18.$$

Conti–Gräf get congruences at least mod 5^4 except for f with $v(\mathcal{L}_f)=-2$, which has no match:

$$f\equiv q+22q^2+11q^3+117q^4+117q^6+92q^7+\cdots \pmod{5^3}.$$

From its q-expansion, $a_2(f)=22$, so that its T_2-characteristic polynomial is

$$x-22=x+103 \mod 5^3.$$

In other words, the mod 5^3 congruence from Expected Theorem excludes precisely this form!